

The University of Chicago

PROBLEMS OF RADIATIVE
TRANSFER IN THE THEORY OF
STELLAR ATMOSPHERES

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF ASTRONOMY AND ASTROPHYSICS
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By
GUIDO MÜNCH

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A THEORETICAL DISCUSSION OF THE CONTINUOUS SPECTRUM OF THE SUN

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ABSTRACT

The observational data relating to the various aspects of the intensity distribution in the continuous spectrum of the sun is examined in the light of recent developments in the theory of radiative equilibrium. It is shown that the intensity distribution in the continuous spectrum at the center of the solar disk, the distribution in the different wave lengths of the emergent monochromatic fluxes, and the law of darkening in the different wave lengths can all be accounted for, within the limitations of the theory, in terms of a single dependence of the continuous absorption coefficient with wave length (see Fig. 1). Further, the required variation of the absorption coefficient in the visual and in the near infrared regions of the spectrum receives a satisfactory explanation in terms of the continuous absorption coefficient of the negative ion of hydrogen.

1. *Introduction.*—In a recent paper S. Chandrasekhar² has rediscussed the problem of radiative transfer in an atmosphere in local thermodynamical equilibrium and has shown that a formula for the temperature distribution of the form

$$T^4 = \frac{3}{4} T_e^4 (\tau + q[\tau]) \quad (1)$$

where T_e denotes the effective temperature and $q(\tau)$ a certain monotonic increasing function of the optical depth τ , is valid also for a nongray atmosphere (in a second approximation) if the mean absorption coefficient $\bar{\kappa}$, in terms of which τ has to be measured, is defined as a straight average of the monochromatic absorption coefficient κ_ν , weighted according to the net flux $F_\nu^{(1)}$ of radiation of frequency ν in a gray atmosphere, and if, further, $\kappa_\lambda/\bar{\kappa}$ is independent of depth. Accordingly, it would appear that, under the stated conditions, the emergent intensity $I_\lambda(\vartheta)$ at the wave length λ and in a direction inclined to the normal by an angle ϑ will be given by

$$I_\lambda(\vartheta) = \int_0^\infty B_\lambda(T_\tau) e^{-\kappa_\lambda \tau \sec \vartheta / \bar{\kappa}} \frac{\kappa_\lambda}{\bar{\kappa}} \sec \vartheta d\tau, \quad (2)$$

where $B_\lambda(T_\tau)$ denotes the Planck function for the temperature T_τ prevailing at the depth τ .

The use of equation (2) for the discussion of the continuous spectrum of the sun and the stars has been well known since Milne's classical paper on "Radiative Equilibrium."³ The method, as applied to the sun, has been to derive $\kappa_\lambda/\bar{\kappa}$ such that equations (1) and (2) together predict the correct intensity distribution in the continuous spectrum. More particularly, attention has been concentrated on two features of this distribution: first, on the intensity distribution in the continuous spectrum as observed at the center of the solar disk and, second, on the law of darkening in the different wave lengths. In this manner two independent determinations of the dependence of the continuous absorption coefficient on the wave length can be made. This is the principle underlying the discussions of Milne,³ Unsöld and Maue,⁴ and Mulders.⁵ But these same discussions seemed to indi-

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² *Ap. J.*, **101**, 328, 1945.

⁴ *Zs. f. Ap.*, **5**, 1, 1932.

³ *Phil. Trans. R. Soc., A*, **223**, 201–255, 1922.

⁵ *Zs. f. Ap.*, **11**, 132, 1935.

cate that the two determinations of κ_λ/κ were in poor agreement. In trying to examine the origin of this disagreement, it appeared that it might be profitable to re-examine the whole problem on the revised basic theory of Chandrasekhar and with a view to correlating the derived variation of κ_λ/κ with the known sources of opacity in the solar atmosphere.

2. *The determination of κ_λ/κ from the intensity distribution of the continuous spectrum at the center of the solar disk.*—At the center of the disk the intensity distribution will be given by

$$I_\lambda(0) = \int_0^\infty B_\lambda(T_\tau) e^{-\kappa_\lambda \tau / \bar{\kappa}} \frac{\kappa_\lambda}{\bar{\kappa}} d\tau, \quad (3)$$

and the determination of $\kappa_\lambda/\bar{\kappa}$ according to this formula requires a knowledge of $I_\lambda(0)$. For the sun this quantity has been derived by Mulders,⁶ who has made special allowances for the energy in the continuous spectrum which has been lost in the absorption lines. Mulders' values were graphically smoothed out, and the specific intensities derived in this manner are given for various wave lengths in Table 1. These values were used in our

TABLE 1
 $\kappa_\lambda/\bar{\kappa}$ DERIVED FROM THE INTENSITIES IN THE CENTER OF THE DISK

$\lambda \text{ \AA}$	$I_\lambda(0) \times 10^{-13}$ (Ergs/Cm ² /Sec)	$\kappa_\lambda/\bar{\kappa}$	$\lambda \text{ \AA}$	$I_\lambda(0) \times 10^{-13}$ (Ergs/Cm ² /Sec)	$\kappa_\lambda/\bar{\kappa}$
3000.....	15.0	1.34	6500.....	29.1	0.82
3230.....	16.8	1.41	6702.....	27.6	0.83
3400.....	19.2	1.38	7000.....	25.6	0.84
3737.....	31.7	0.94	7500.....	22.0	0.86
3800.....	36.9	0.80	8000.....	19.0	0.98
3900.....	42.0	0.72	8500.....	16.7	1.00
4000.....	44.0	0.67	8580.....	16.4	1.02
4265.....	46.2	0.65	9000.....	14.3	1.12
4500.....	45.4	0.66	9500.....	12.8	1.09
5000.....	41.6	0.72	10000.....	11.7	1.03
5062.....	41.1	0.72	10080.....	11.5	0.99
5500.....	37.6	0.73	10500.....	10.7	0.91
5955.....	33.7	0.77	11000.....	9.7	0.79
6000.....	33.1	0.78			

further discussion.

In deriving κ_λ/κ according to equation (3), the temperature distribution was assumed to be given by its solution in the "first approximation," namely,

$$T^4 = \frac{3}{4} T_e^4 \left(\tau + \frac{2}{3} \right) = \sqrt{3} T_0^4 \left(\tau + \frac{2}{3} \right). \quad (4)$$

It will be noticed that in equation (4) we have used the Hopf-Bronstein relation between the boundary and the effective temperatures. For the temperature distribution given by equation (4) the emergent intensity $I_\lambda(0)$ at the center of the disk can be expressed in the form

$$I_\lambda(0) = 2hc^2 \left(\frac{2^{1/4}}{3^{1/8}} \frac{k}{hc} \right)^5 T_0^5 f(\gamma, p), \quad (5)$$

where

$$\gamma = \frac{3^{1/8}}{2^{1/4}} \frac{hc}{k\lambda T_0} \quad \text{and} \quad p = \frac{3}{2} \frac{\bar{\kappa}}{\kappa_\lambda} \quad (6)$$

⁶ Dissertation, Utrecht, 1934.

and

$$f(\gamma, p) = \gamma^5 \int_0^\infty \frac{e^{-t}}{e^{\gamma(1+pt)^{-1/4}} - 1} dt \quad (7)$$

is the function which has been tabulated by Milne⁷ and Lindblad for various values of the arguments. (In equations [5] and [6] h , c , and k have their usual meanings.)

For the effective temperature T_e of the sun the value 5740° K was adopted, which is the value derived by Milne⁸ and Minnaert⁹ in their reductions of Abbot's measures to the absolute scale.¹⁰ With this value of T_e the boundary temperature T_0 is 4656° K, and equation (5) reduces to

$$I_\lambda(0) = 5.057 \times 10^{12} f(\gamma, p). \quad (8)$$

With the intensities given in Table 1 and with the tables of the function $f(\gamma, p)$ provided by Milne and Lindblad it is a simple matter to derive $\kappa_\lambda/\bar{\kappa}$ for the various wave lengths.

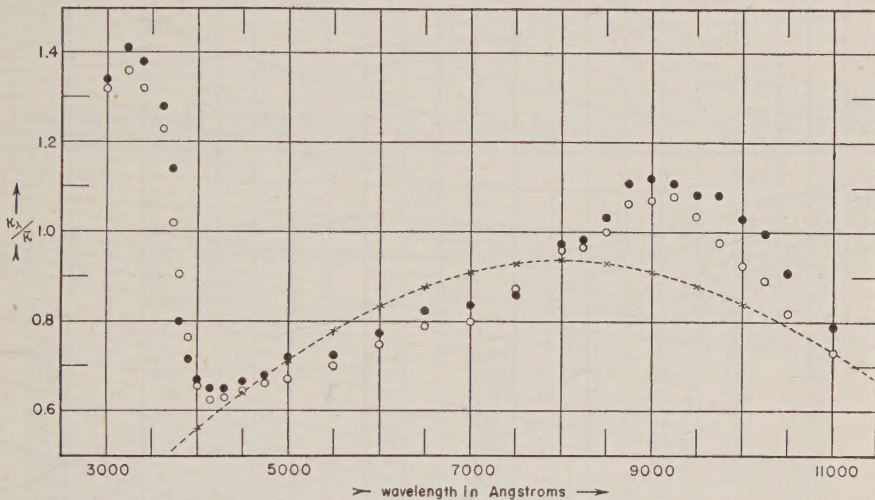


FIG. 1.—The dependence of $\kappa_\lambda/\bar{\kappa}$ on wave length. The solid circles represent values obtained from the intensities in the center of the solar disk; the open circles, those obtained from the monochromatic fluxes. The dotted line is the absorption of H^- computed on the assumption that this ion is the only source of opacity between 4000 Å and 16,000 Å.

The results are given in Table 1 and are further illustrated in Figure 1. The dependence of the continuous absorption on wave length which we have derived is in general agreement with that found earlier by Milne and Mulders. However, it differs in details from the earlier determinations for the following reasons. In Milne's discussion the effect of the absorption lines on the observed continuous spectrum was not allowed for, while in Mulders' discussion a boundary temperature, T_0 , allowing for the so-called "blanketing effect," was used. On this latter point, reference may be made to a recent paper by Unsöld,¹¹ who has questioned the use of a blanketing theory in this form.

⁷ *Op. cit.*, Appen. I.

⁸ *Ibid.*, Appen. II.

⁹ *B.A.N.*, 51, 75, 1924.

¹⁰ It should, perhaps, be mentioned in this context that Mulders' discussion (from which the values of $I_\lambda(0)$ given in Table 1 have been taken) is also based on these same measures of Abbot.

¹¹ *Zs. f. Ap.*, 22, 356, 1943.

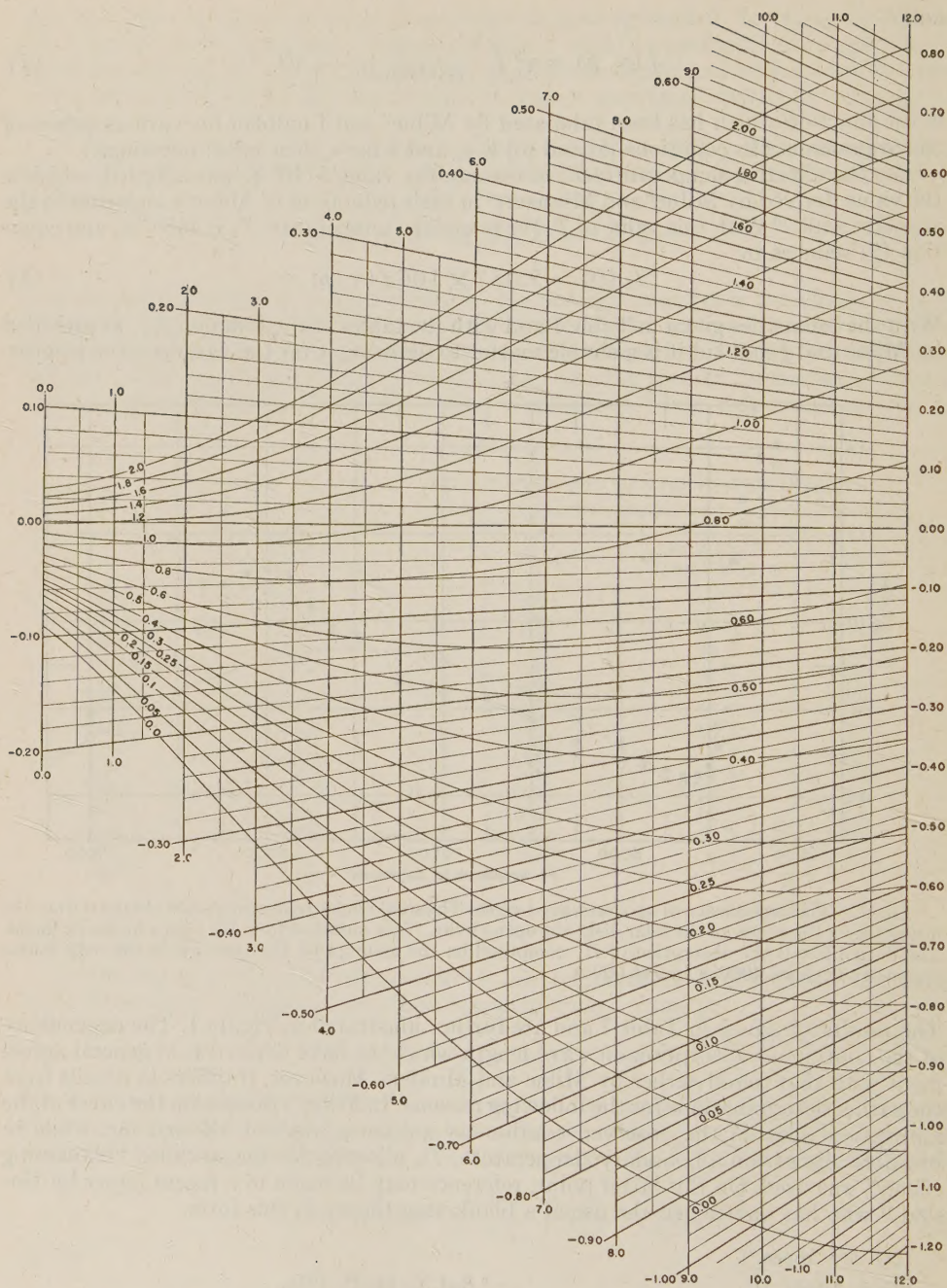


FIG. 2.—Graphical representation of Burkhardt's table (*Zs. f. Ap.*, 13, 56, 1936). Every point of the region represented corresponds to one set of values $(\alpha, \log F_\lambda(0)/B_\lambda(T_e), \bar{\kappa}/\kappa_\lambda)$. The vertical straight lines are the loci of constant α ; the diagonal straight lines, the loci of constant $F_\lambda(0)/B_\lambda(T_e)$; and the curves themselves are the loci of constant $\bar{\kappa}/\kappa_\lambda$.

3. *The determination of $\kappa_\lambda/\bar{\kappa}$ from the intensity distribution in the emergent flux.*—A determination of $\kappa_\lambda/\bar{\kappa}$ can also be made from the intensity distribution in the emergent flux. For on the assumptions under which equation (1) for the temperature distribution may be expected to be approximately valid for a nongray atmosphere, it is true that¹²

$$F_\lambda(0) = 2 \int_0^\infty B_\lambda(T_\tau) E_2\left(\frac{\kappa_\lambda}{\bar{\kappa}} \tau\right) \frac{\kappa_\lambda}{\bar{\kappa}} d\tau, \quad (9)$$

where

$$E_2(x) = \int_1^\infty \frac{e^{-xw}}{w^2} dw. \quad (10)$$

Equation (9) can be expressed alternatively in the form

$$\log \frac{F_\lambda(0)}{B_\lambda(T_e)} = \log \left\{ 2(e^\alpha - 1) \int_0^\infty \frac{E_2(\kappa_\lambda \tau / \bar{\kappa})}{e^{\alpha[3/4(\tau + q\tau^2)] - 1/4} - 1} \frac{\kappa_\lambda}{\bar{\kappa}} d\tau \right\}, \quad (11)$$

where

$$\alpha = \frac{hc}{k\lambda T_e}. \quad (12)$$

The expression on the right-hand side of (11) has been evaluated by Burkhardt¹³ for various values of $\kappa_\lambda/\bar{\kappa}$ and α . It was found that, in practice, the use of Burkhardt's tables

TABLE 2
 $\kappa_\lambda/\bar{\kappa}$ DERIVED FROM THE MONOCHROMATIC FLUXES

λA	$F_\lambda(0) \times 10^{-13}$ (Ergs/Cm ² /Sec)	$\kappa_\lambda/\bar{\kappa}$	λA	$F_\lambda(0) \times 10^{-13}$ (Ergs/Cm ² /Sec)	$\kappa_\lambda/\bar{\kappa}$
3000.....	9.9	1.32	6500.....	24.3	0.79
3230.....	11.7	1.36	7000.....	21.3	0.80
3400.....	13.6	1.32	7500.....	18.4	0.87
3737.....	20.3	1.02	8000.....	15.9	0.96
3800.....	23.4	0.91	8500.....	14.0	1.00
3900.....	28.5	0.76	9000.....	12.2	1.07
4000.....	32.6	0.66	9500.....	11.0	1.04
4500.....	34.2	0.64	10000.....	10.2	0.93
5000.....	32.2	0.67	10500.....	9.5	0.82
5500.....	29.9	0.70	11000.....	8.8	0.73
6000.....	27.2	0.75			

is enormously facilitated by plotting them in the manner of Figure 2. With this plot, graphical interpolation, giving sufficient accuracy, is possible with considerable ease.

To determine $\kappa_\lambda/\bar{\kappa}$, according to equation (11), we require, of course, a knowledge of $F_\lambda(0)$ for various wave lengths. These follow also from Mulders' discussion and have been tabulated by Unsöld.¹⁴ The values of $F_\lambda(0)$ used in the present discussion are those given in Table 2; these were obtained by graphical interpolation from those given by Unsöld. It may be pointed out in this connection that the observational determination of $F_\lambda(0)$ follows from a more direct and straightforward procedure than does $I_\lambda(0)$ (see § 4, below) and is, accordingly, more trustworthy.

With the values of $F_\lambda(0)$ given in Table 2 and with the plot of Figure 2, $\kappa_\lambda/\bar{\kappa}$ was de-

¹² Cf. E. A. Milne, *Handb. d. Ap.*, 3, Part I, 147-155, Berlin: Springer, 1930.

¹³ *Zs. f. Ap.*, 13, 56, 1936.

¹⁴ *Physik der Sternatmosphären*, Tables 7 and 8, pp. 38-39, Berlin: Springer, 1938.

rived for various wave lengths. The results are given in Table 2. For comparison with the values derived from the intensities in the center, the determination of $\kappa_\lambda/\bar{\kappa}$ from the monochromatic fluxes is also plotted in Figure 1. The agreement between the two determinations is seen to be very satisfactory, indeed; and it clearly implies that the same variation of the absorption coefficient with wave length is adequate (within limits) for accounting both for the law of darkening and for the intensity distribution over the different wave lengths (see § 4, below). Finally, we may draw attention to the fact that observational data relating to the monochromatic fluxes are available up to wave length λ 23,480 Å. But, unfortunately, Burkhardt's table is not extensive enough to use this data for the evaluation of $\kappa_\lambda/\bar{\kappa}$ in these regions of the spectrum. However, the indications are that $\kappa_\lambda/\bar{\kappa}$ has a deep minimum at about λ 16,000 Å.

4. *The darkening in the different wave lengths.*—We shall begin our discussion of this phenomenon by briefly recapitulating the procedure which is followed in the observational determination of the quantities $I_\lambda(0)$ and $F_\lambda(0)$ in absolute measure. Essentially what the observations give directly are the quantities $f_\lambda = c_1 F_\lambda(0)$ and $i_\lambda = c_2 I_\lambda(0)$, where c_1 and c_2 are certain scale factors to be determined, and the function

$$\phi_\lambda(\vartheta) = \frac{i_\lambda(\vartheta)}{i_\lambda(0)} = \frac{I_\lambda(\vartheta)}{I_\lambda(0)}. \quad (13)$$

In other words, the intensities at the center and the monochromatic fluxes are known in different units, and the law of darkening is also known. The constant c_1 is determined from the condition

$$\frac{1}{c_1} \int_0^\infty f_\lambda d\lambda = \int_0^\infty F_\lambda(0) d\lambda = F; \quad (14)$$

and as F is simply related to the solar constant, the unit in which f_λ measures the monochromatic fluxes is determinate.

The constant c_2 is next determined from the condition

$$\frac{2}{c_2} \int_0^{\pi/2} i_\lambda(0) \phi_\lambda(\vartheta) \sin \vartheta \cos \vartheta d\vartheta = F_\lambda. \quad (15)$$

It is now apparent that the substantial agreement between the two determinations of $\kappa_\lambda/\bar{\kappa}$, from the intensities at the center of the disk and from the fluxes, must mean that the same dependence of the absorption coefficient on wave length accounts for the law of darkening satisfactorily. That this is actually the case can be shown by evaluating $I_\lambda(\vartheta)$ according to equation (2), using the same values of $\kappa_\lambda/\bar{\kappa}$ as has been derived from the intensity distribution at the center of the disk. To do this we re-write equation (2) in a form analogous to (8). We have

$$I_\lambda(\vartheta) = 5.057 \times 10^{12} f \left(\gamma, \frac{3}{2} \frac{\bar{\kappa}}{\kappa_\lambda} \cos \vartheta \right). \quad (16)$$

Using this formula and the Milne-Lindblad tabulation of the function $f(\gamma, p)$, $I_\lambda(\vartheta)/I_\lambda(0)$ was evaluated for various wave lengths. The results are exhibited in Figure 3, together with the darkening observations as collected by Unsöld.¹⁵ It is seen that, as expected, the agreement between the observed and the computed darkening is quite good. However, there are systematic differences, in the sense that toward the limb there is observationally less contrast than computed. But it should be pointed out that the foundations both of the theory and of observations, in so far as they refer to the extreme limb, are not so secure. On the observational side, the photometric problem is one of exceptional complex-

¹⁵ *Ibid.*, pp. 34–35. Tables 3 and 4.

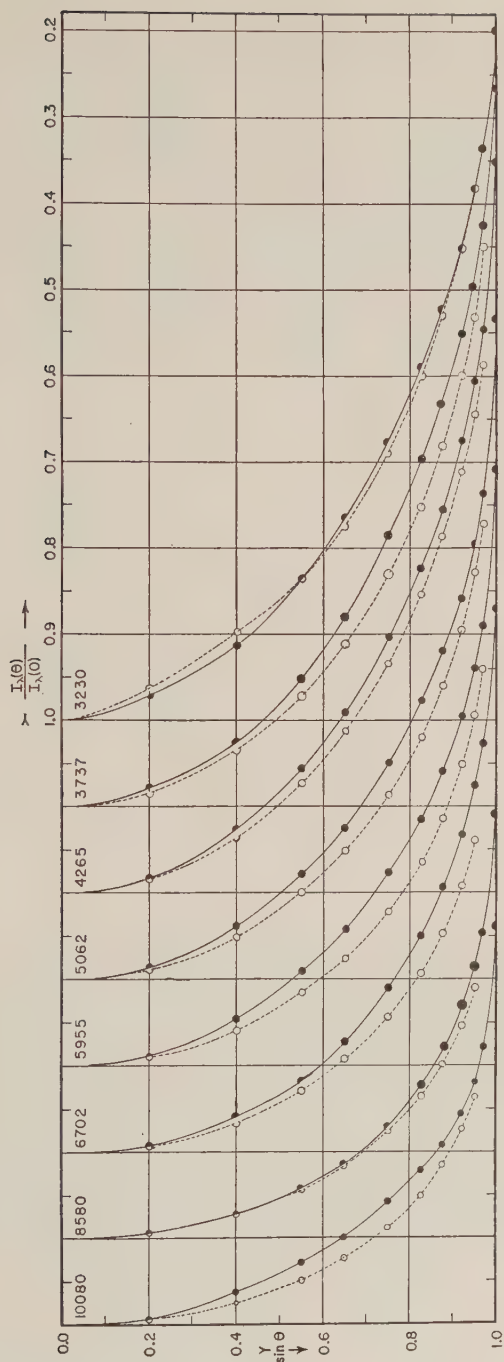


FIG. 3.—Comparison of the observed and computed darkening. Open circles are observational values; solid circles, the theoretical ones. The scale of the abscissae is displaced 0.10 to the left at each wave length considered.

ity, in view of the difficulties associated with the elimination of scattering not only from the earth's atmosphere but from the instruments employed. These factors become major sources of error for the observations at the limb. On the theoretical side the intensities at the limb depend very sensitively on the boundary temperature, and it is well known that this can be affected in a variety of ways. In this connection it may be noted that, for a nongray atmosphere, the Hopf-Bronstein relation between the effective and the boundary temperatures is not strictly true.¹⁶ It would be of interest to examine the consequences of this correction in the first instance.

With reference to the remarks made toward the end of the last paragraph, we should draw further attention to the general unreliability of determining $\kappa_\lambda/\bar{\kappa}$ from the law of darkening alone. This is clear, for example, from Figure 4, in which we have illustrated the darkening function $\phi_\lambda(\vartheta)$ for $\lambda = 3230$ Å for various values of ϑ as a function of $\kappa_\lambda/\bar{\kappa}$. It is seen that a change of 10 per cent in $\kappa_\lambda/\bar{\kappa}$ produces less than 2 per cent change in $\phi_\lambda(\vartheta)$ at the limb; at other points on the disk the change in $\phi_\lambda(\vartheta)$ is much less. It would

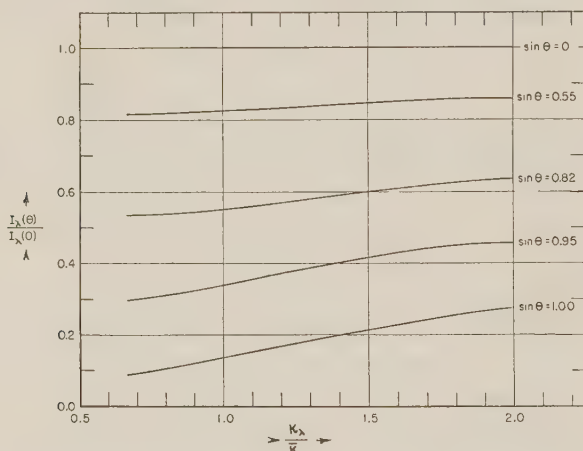


FIG. 4.—The dependence of the darkening function on $\kappa_\lambda/\bar{\kappa}$ for $\lambda = 3230$ Å at different points of the solar disk.

therefore seem that the estimation of $\kappa_\lambda/\bar{\kappa}$ from the center-limb contrast is particularly sensitive to errors in observations and is accordingly not to be recommended.

5. *The continuous absorption coefficient.*—The dependence of the continuous absorption coefficient on wave length shown in Figure 1 is such as to suggest that we should perhaps distinguish between three principal sources of opacity in the solar atmosphere, contributing, respectively, in the regions $\lambda < 4000$ Å, 4000 Å $< \lambda < 16,000$ Å, and $\lambda > 16,000$ Å. And from our present knowledge of the physical conditions in the photosphere of the sun it is not difficult to account for this feature in a general way. In the region to the violet of 4000 Å, the contribution to the opacity must come from the photoelectric ionization of the "metals." It is further likely that, on account of the excessive crowding of the absorption lines in this region, there is also an effective increase of the continuous absorption coefficient. In the region 4000 Å $< \lambda < 16,000$ Å it is fairly certain that the negative hydrogen ions provide the principal source of opacity,¹⁷ while beyond 16,000 Å the free-free transitions in the field of the hydrogen atom must begin to be important.

¹⁶ Cf. Chandrasekhar, *op. cit.*, eqs. (109) and (119), for the relation valid in a higher approximation.

¹⁷ Wildt, *A p. J.*, 89, 295, 1939. Also B. Strömgren, *Festschrift für Elis Strömgren*, p. 218, Copenhagen, 1940.

By considering the continuous absorption in the region $4000 \text{ \AA} < \lambda < 16,000 \text{ \AA}$ in greater detail, it is possible to make a more specific comparison with the physical theory. Chandrasekhar¹⁸ has recently made what appears to be a fairly definitive evaluation of

TABLE 3
THE MEAN ABSORPTION COEFFICIENT OF H^-

τ	0.0	0.2	0.5	1.0	2.0
$\bar{\kappa}(H^-) \times 10^{17} \text{ cm}^{-2}$	2.82	2.90	2.94	2.89	2.76

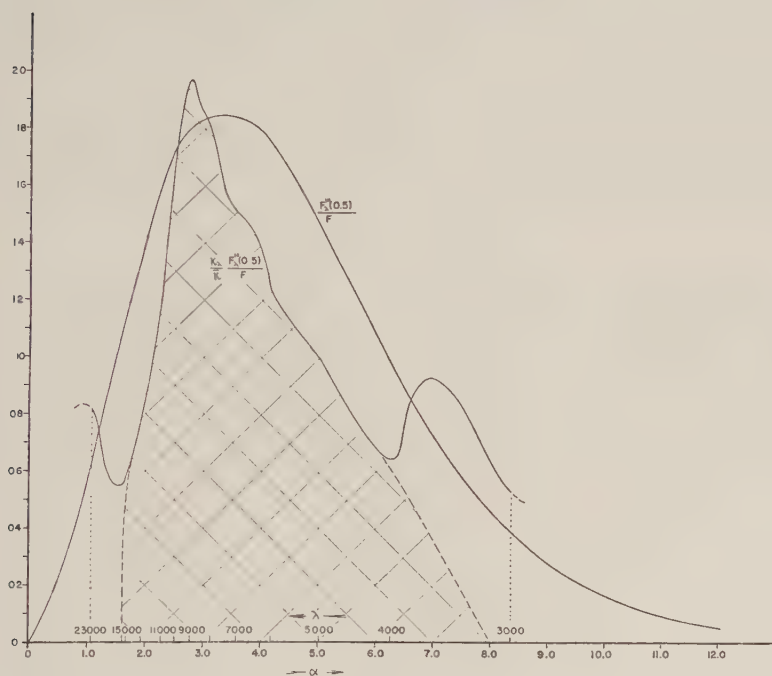


FIG. 5.—The net monochromatic flux $F_{\lambda}^{(1)}$ in units of F at $\tau = 0.5$ and $\kappa_{\lambda} F_{\lambda}^{(1)} / \bar{\kappa}$. The shaded portion is assumed to be the contribution of absorption by H^- to the integral $\int \kappa_{\lambda} F_{\lambda}^{(1)} / \bar{\kappa}$. (The abscissae is $\alpha = hc/k\lambda T_e$.)

the continuous absorption coefficient $\kappa_{\lambda}(H^-)$ of the negative hydrogen ion. Using these values, we can evaluate the mean absorption coefficient $\bar{\kappa}(H^-)$ arising from this source, according to the formula¹⁹

$$\bar{\kappa}(H^-; \tau) = \frac{1}{F} \int_0^{\infty} \kappa_{\lambda}(H^-) F_{\lambda}^{(1)}(\tau) d\lambda, \quad (17)$$

¹⁸ *A p. J.*, **102**, 395, 1945.

¹⁹ S. Chandrasekhar, *A p. J.*, **101**, 328, eq. (42), 1945.

where $F_{\lambda}^{(1)}(\tau)$ denotes the monochromatic flux at wave length λ and depth τ . By using for $F_{\lambda}^{(1)}(\tau)$ the values tabulated in Chandrasekhar's paper, $\kappa(H^-)$ was evaluated at various depths. The results are summarized in Table 3, where it is seen that the dependence of $\kappa(H^-)$ on the depth is very slight. The assumption of the constancy of $\kappa_{\lambda}/\bar{\kappa}$ arising from this source can therefore be justified. It would further appear that averaging κ_{λ} with $F_a^{(1)}(\tau=0.5)$ ($a = h\nu/kT$) would be sufficient. Using the mean of our two determinations of $\kappa_{\lambda}/\bar{\kappa}$, we have plotted the quantity $\kappa_{\lambda}F_{\lambda}^{(1)}(\tau=0.5)/\bar{\kappa}$ in Figure 5. Now the theory would require that

$$\frac{1}{F} \int_0^{\infty} \frac{\kappa_{\lambda} F_{\lambda}^{(1)}}{\bar{\kappa}} d\lambda = 1. \quad (18)$$

Actually, the evaluation of the integral on the left-hand side between $\lambda = 3000$ Å and $\lambda = 23,000$ Å yields the value 0.76. Considering that this evaluation does not include the contribution of the spectral region to the violet of 3000 Å, we find this result to be satisfactory. The requirement that the integral over the entire wave-length range should equal unity would seem to suggest that $\kappa_{\lambda}/\bar{\kappa}$ has values in the neighborhood of 2 far into the violet.

Finally, assuming that the shaded portion in Figure 5 arises from H^- alone, it is found that the contribution of H^- to κ amounts to 61 per cent. Accordingly, we can compare the derived values of $\kappa_{\lambda}/\bar{\kappa}$ in the region $4000 \text{ Å} < \lambda < 16,000 \text{ Å}$ with the values

$$\frac{0.61}{2.94} \times 10^{17} \kappa_{\lambda}(H^-). \quad (19)$$

Figure 1 shows the comparison between both sets of values, the dotted curve representing expression (19) computed with Chandrasekhar's values of $\kappa_{\lambda}(H^-)$. As one can see, the agreement is satisfactory.

The writer is deeply indebted to Dr. S. Chandrasekhar for his constant encouragement and advice during the time this work was carried out.

THE EFFECT OF THE ABSORPTION LINES ON THE TEMPERATURE DISTRIBUTION OF THE SOLAR ATMOSPHERE

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ABSTRACT

In this paper the influence of the absorption lines on the temperature distribution of a stellar atmosphere is reconsidered, using the new methods in the theory of radiative equilibrium. The analysis is carried out by solving simultaneously the appropriate equations of transfer for the continuous and the line spectra, under the condition of a constant net flux and on the assumption that the probability of occurrence of an absorption line at a frequency ν is governed by a distribution function $w_2(\nu)$. When $w_2(\nu)$ is independent of frequency, several cases are investigated according to whether the lines arise in absorption and/or scattering. When $w_2(\nu)$ is not constant throughout the spectrum, only the case of pure local thermodynamical equilibrium is considered. For the various cases considered, numerical models are computed in the first two approximations. These models are characterized by an emergent flux in the lines of the same order of magnitude as that observed in the solar spectrum. From the discussion of the different models, a value for the boundary temperature of the sun near 4600° K is derived. Further, it is shown that the darkening in integrated light of the solar disk is accounted for in terms of the combined darkening in the lines and in the continuum of one of the models considered. Finally, some remarks on the phenomenon of monochromatic darkening are also made.

1. *Introduction.*—The effect of the absorption lines on the temperature distribution in a stellar atmosphere has been studied in the past, mainly with the purpose of explaining the observed features of the continuous spectrum of the sun which could not be explained by the simple theory of radiative transfer in a gray atmosphere. As the writer has recently shown,² the dependence on wave length of the continuous absorption coefficient of the negative hydrogen ion, together with the theory of a nongray stellar atmosphere (disregarding, however, the presence of lines), accounts for the main features of the departures from grayness of the solar continuum, except in the ultraviolet part of the spectrum. In the latter region the problem is complicated by the overcrowding of the Fraunhofer lines, which produces a total absorption of an unknown amount. Nevertheless, when the darkening in the nongray atmosphere is computed and the comparison with the observed darkening is carried out, small systematic differences are found, in the sense that the computed center-limb contrast is higher than the observed one. It would seem that, to account for these differences, we should allow for the presence of absorption lines in determining the temperature distribution. In this connection we may recall that earlier treatments of this "blanketing effect" by E. A. Milne³ and S. Chandrasekhar⁴ have shown that the modifications of the temperature distribution in a gray atmosphere are not inappreciable.

The present paper presents some further developments relating to the effect produced by the absorption lines on the temperature distribution of a stellar atmosphere in radiative equilibrium, on the basis of the methods recently developed by S. Chandrasekhar,⁵ in connection with a variety of problems of radiative transfer. These methods

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² *Ap. J.*, **102**, 385, 1945.

³ *Phil. Trans. R. Soc., A*, **223**, 201–255, 1922; *Observatory*, **51**, 88, 1928.

⁴ *M.N.*, **96**, 21, 1936. This paper will be referred to hereafter as "paper B." See also V. Ambarzumian, *Pub. Astr. Obs. U. Leningrad*, new ser., **1**, 7, 1936.

⁵ *Ap. J.*, **100**, 76, 1944, and **101**, 328, 1945. These papers will be referred to as papers "II" and "VII," respectively.

enable us to treat the problem not only in a more exact fashion but also under less restrictive conditions.

2. *The equations of transfer and their approximate forms.*—The basic idea involved in the precise formulation of the problem was given first in paper B. Let $I_\nu(x, \mu)$ denote, as usual, the intensity of radiation of frequency ν and in a direction forming an angle ϑ ($= \cos^{-1} \mu$) with the positive normal, $B_\nu(T_x)$ the Planck function for the temperature T_x prevailing at the depth x , and κ_ν the continuous absorption coefficient. The equation of transfer for the continuous spectrum, on the hypothesis of local thermodynamic equilibrium, is

$$\mu \frac{dI_\nu}{\rho dx} = -\kappa_\nu (I_\nu - B_\nu). \quad (1)$$

On the other hand, for points within the line spectrum (assuming that for our purposes all lines can be treated in the same way as resonance lines), the appropriate equation of transfer is

$$\mu \frac{dI_\nu}{\rho dx} = -(\kappa_\nu + a_\nu) I_\nu + (1 - \epsilon_\nu) a_\nu J_\nu + (\kappa_\nu + \epsilon_\nu a_\nu) B_\nu \quad (2)$$

where

$$J_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu(x, \mu) d\mu, \quad (3)$$

and, further, where a_ν is the mean coefficient of scattering in the line and ϵ_ν the probability that the radiation of frequency ν is lost by photoelectric ionizations and collisions of the second kind. If we remember that a_ν and ϵ_ν in a stellar spectrum are highly singular functions of frequency, it would seem that the adoption of constant values a and ϵ for them, in the domain covered by the line spectrum, is likely to have a statistical significance. Without restricting the generality of the analysis as much as might appear at first sight, we can also suppose κ_ν to be a constant independent of frequency. In this context we may recall that, in paper VII, Chandrasekhar found that the temperature distribution in a gray atmosphere coincides with the temperature distribution in a non-gray atmosphere "up to a second approximation," provided that the optical depth is measured in terms of a certain appropriately defined mean $\bar{\kappa}$.

We now introduce two distribution functions $w_1(\nu)$ and $w_2(\nu) = 1 - w_1(\nu)$, that govern the probability of our finding at the frequency ν the continuous and the line spectra, respectively. The integration of equations (1) and (2) over all frequencies then gives

$$\mu \frac{dI^{(1)}}{d\tau} = I^{(1)} - \int_0^\infty w_1(\nu) B_\nu d\nu \quad (4)$$

and

$$\mu \frac{dI^{(2)}}{d\tau} = (1 + \xi) I^{(2)} - (1 - \epsilon) \xi J^{(2)} - (1 + \epsilon \xi) \int_0^\infty w_2(\nu) B_\nu d\nu, \quad (5)$$

where τ is the optical depth measured in terms of the constant coefficient of continuous absorption and $\xi = a/\kappa$. It is to be noticed that in equations (4) and (5) we have introduced the superscripts (1) and (2) to distinguish the quantities related to the continuous and to the line spectra, respectively.

Equations (4) and (5) must be solved simultaneously under the condition of constant flux, expressed by the relation

$$J^{(1)} + (1 + \epsilon \xi) J^{(2)} = \int_0^\infty [w_1(\nu) + (1 + \epsilon \xi) w_2(\nu)] B_\nu d\nu. \quad (6)$$

In the following sections we shall be concerned with the solution of the system of equations (4), (5), and (6) under different conditions: First, we shall consider the cases

when $w_1(\nu)$ and $w_2(\nu)$ are independent of frequency, that is, when the lines can be considered uniformly distributed throughout the spectrum. In this case, equations (4), (5), and (6) reduce to

$$\mu \frac{dI^{(1)}}{d\tau} = I^{(1)} - w_1 B, \quad (7)$$

$$\mu \frac{dI^{(2)}}{d\tau} = (1 + \xi) I^{(2)} - (1 - \epsilon) \xi J^{(2)} - (1 + \epsilon \xi) w_2 B, \quad (8)$$

and

$$J^{(1)} + (1 + \epsilon \xi) J^{(2)} = [w_1 + (1 + \epsilon \xi) w_2] B. \quad (9)$$

These equations have been solved by Chandrasekhar in paper B, following a Milne-Eddington type of approximation.

3.1. *The absorption lines formed in pure local thermodynamic equilibrium.*—When the Fraunhofer lines arise in a process of pure absorption, $\epsilon = 1$ and equations (7) and (8) become

$$\left. \begin{aligned} \mu \frac{dI^{(1)}}{d\tau} &= I^{(1)} - w_1 B, \\ \frac{\mu}{\gamma} \frac{dI^{(2)}}{d\tau} &= I^{(2)} - w_2 B, \end{aligned} \right\} \quad (10)$$

where $\gamma = 1 + \xi$. An important contribution to the study of this case has been given by E. Hopf,⁶ who was able to evaluate the exact value of the integrated Planck function $B(\tau)$ at $\tau = 0$. Following this latter work, we shall formally generalize equations (10) to the system

$$\frac{\mu}{\gamma_a} \frac{dI^{(a)}}{d\tau} = I^{(a)} - w_a B \quad (a = 1, 2, \dots, m), \quad (11)$$

which represents the situation when the spectrum is divided into m parts, each characterized by a parameter γ_a , supposed to be independent of depth, and a relative density w_a . The condition of radiative equilibrium, under these more general circumstances, is

$$B(\tau) \Sigma \gamma_a w_a = \Sigma \gamma_a J^{(a)} = \Sigma \gamma_a \frac{1}{2} \int_{-1}^{+1} I^{(a)}(\tau, \mu) d\mu. \quad (12)$$

Using the foregoing relation, we can re-write equations (11) in the form

$$\frac{\mu}{\gamma_a} \frac{dI^{(a)}}{d\tau} = I^{(a)} - w_a \frac{\Sigma \gamma_\beta J^{(\beta)}}{\Sigma \gamma_\beta w_\beta} \quad (a = 1, 2, \dots, m). \quad (13)$$

Following the method developed in paper II, this system is replaced in the n th approximation by $2nm$ linear equations obtained by expressing $J^{(\beta)}$ in terms of Gauss's quadrature formula, involving the zeros μ_i of the $2n$ th Legendre polynomial and the associated weights a_i . Thus we get

$$\frac{\mu_i}{\gamma_a} \frac{dI_i^{(a)}}{d\tau} = I_i^{(a)} - \frac{w_a}{\Sigma \gamma_\beta w_\beta} \sum_{\beta=1}^m \gamma_\beta \frac{1}{2} \sum_{j=-n}^n a_j I_j^{(\beta)} \quad \left(\begin{matrix} i = \pm 1, \dots, \pm n \\ a = 1, 2, \dots, m \end{matrix} \right), \quad (14)$$

where $I_i^{(a)} = I^{(a)}(\tau, \mu_i)$. This system of linear equations admits solutions of the form

$$I_i^{(a)} = \text{constant} \frac{e^{-k\tau}}{1 + k\mu_i/\gamma_a}, \quad (15)$$

where k_a is any of the $(2nm - 2)$ nonvanishing roots of the equation

$$2 \sum_{a=1}^m \gamma_a w_a = \sum_{a=1}^m \gamma_a w_a \sum_{j=-n}^n \frac{a_j}{1 + k \mu_j / \gamma_a}. \quad (16)$$

Equations (14) also admit a solution of the form

$$I_i^{(a)} = b w_a \left(\tau + \frac{\mu_i}{\gamma_a} + Q \right), \quad (17)$$

where b and Q are constants. Therefore, the general solution is

$$I_i^{(a)} = b w_a \left\{ \tau + \frac{\mu_i}{\gamma_a} + Q + \sum_{\beta=1}^{nm-1} \frac{L_\beta e^{-k_\beta \tau}}{1 + k_\beta \mu_i / \gamma_a} + \sum_{\beta=1}^{nm-1} \frac{L_{-\beta} e^{k_\beta \tau}}{1 - k_\beta \mu_i / \gamma_a} \right\} \quad (18)$$

$(\alpha = 1, 2, \dots, m; i = \pm 1, \pm 2, \dots, \pm n).$

The integration constants L_β and $L_{-\beta}$ ($\beta = 1, \dots, nm - 1$), b and Q , are fixed by the boundary conditions. First, we require that the solution be asymptotically linear in τ as $\tau \rightarrow \infty$; and, second, we make use of the fact that there is no radiation incident at $\tau = 0$. The first condition implies that $L_{-\beta} = 0$ for every β , while the second condition gives $I_{-i}^{(a)}(0) = 0$, for every a and i . Thus

$$\sum_{\beta=1}^{nm-1} \frac{L_\beta}{1 - k_\beta \mu_i / \gamma_a} + Q = \frac{\mu_i}{\gamma_a} \quad \left(\begin{array}{l} i = 1, \dots, n \\ \alpha = 1, \dots, m \end{array} \right). \quad (19)$$

These are the nm equations which determine the $(nm - 1)$ L_β 's, and Q . The further integration constant, b , is readily found to be related to the constant net flux, according to

$$b = \frac{3}{4} F \frac{1}{\sum w_a / \gamma_a}. \quad (20)$$

The solution which satisfies the boundary conditions must, accordingly, be of the form

$$I_i^{(a)} = \frac{3}{4} \frac{F w_a}{\sum w_a / \gamma_a} \left\{ \tau + \frac{\mu_i}{\gamma_a} + Q + \sum_{\beta=1}^{nm-1} \frac{L_\beta e^{-k_\beta \tau}}{1 + k_\beta \mu_i / \gamma_a} \right\}. \quad (21)$$

Using the foregoing solution, we can find $J^{(a)}$ and $F^{(a)}$ by representing the integrals by the corresponding Gaussian sums. Thus

$$J^{(a)} = \frac{3}{4} \frac{F w_a}{\sum w_a / \gamma_a} \left\{ \tau + Q + \sum_{\beta=1}^{nm-1} L_\beta e^{-k_\beta \tau} \sum_{j=1}^n \frac{a_j}{1 - \mu_j^2 k_\beta^2 / \gamma_a^2} \right\}, \quad (22)$$

and

$$F^{(a)} = \frac{3}{2} \frac{F w_a}{\sum w_a / \gamma_a} \left\{ \frac{2}{3} \frac{1}{\gamma_a} + 2 \gamma_a \sum_{\beta=1}^{nm-1} \frac{L_\beta}{k_\beta} e^{-k_\beta \tau} \left(1 - \sum_{j=1}^n \frac{a_j}{1 - \mu_j^2 k_\beta^2 / \gamma_a^2} \right) \right\}. \quad (23)$$

The integrated Planck function is then obtained by substituting equation (22) in equation (12). We find

$$B(\tau) = \frac{3}{4} \frac{F}{\sum w_a / \gamma_a} \left\{ \tau + Q + \sum_{\beta=1}^{nm-1} L_\beta e^{-k_\beta \tau} \right\}. \quad (24)$$

3.2. *The value of the boundary temperature.*—We shall now show how the value of $B(\tau)$ at $\tau = 0$ can be evaluated explicitly. Consider the function

$$S(\mu) = \sum_{\beta=1}^{nm-1} \frac{L_{\beta}}{1 - \mu k_{\beta}} + Q - \mu. \quad (25)$$

According to equation (19), $S(\mu_i/\gamma_a) = 0$ for every a and i . Hence, multiplying equation (25) by

$$R(\mu) = \prod_{\beta=1}^{nm-1} (1 - \mu k_{\beta}), \quad (26)$$

we obtain a polynomial of degree nm in μ , which vanishes for $\mu = \mu_i/\gamma_a$ ($i = 1, 2, \dots, n$; $a = 1, 2, \dots, m$). Accordingly, $S(\mu)R(\mu)$ cannot differ from the polynomial

$$P(\mu) = \prod_{\beta=1}^m \prod_{i=1}^n \left(\mu - \frac{\mu_i}{k_{\beta}} \right) \quad (27)$$

by more than a constant factor. This factor can be determined by comparing the coefficients of μ^{nm} in the expressions for $R(\mu)S(\mu)$ and $P(\mu)$. In the former it is $(-1)^{nm} k_1 k_2 \dots k_{nm-1}$, while in the latter it is unity. Hence

$$S(\mu) = (-1)^{nm} k_1 k_2 \dots k_{nm-1} \frac{P(\mu)}{R(\mu)}. \quad (28)$$

Setting $\mu = 0$ in the foregoing equation, we obtain

$$S(0) = k_1 k_2 \dots k_{nm-1} \frac{(\mu_1 \mu_2 \dots \mu_n)^m}{(\gamma_1 \gamma_2 \dots \gamma_m)^n}. \quad (29)$$

In this expression the product of the characteristic roots appears; this can be evaluated by writing down the characteristic equation (16) in the form of a polynomial in k^2 . We find

$$k_1 k_2 \dots k_{nm-1} = \frac{(\gamma_1 \gamma_2 \dots \gamma_m)^n}{(\mu_1 \mu_2 \dots \mu_n)^m} \left(\frac{\sum w_a / \gamma_a}{3 \sum w_a \gamma_a} \right)^{\frac{1}{2}}, \quad (30)$$

and equation (29) becomes

$$S(0) = \left(\frac{\sum w_a / \gamma_a}{3 \sum w_a \gamma_a} \right)^{\frac{1}{2}}. \quad (31)$$

On the other hand, according to equations (24) and (25),

$$B(0) = \frac{3}{4} F \frac{S(0)}{\sum w_a / \gamma_a}; \quad (32)$$

or, using equation (31), we have

$$B(0) = \frac{\sqrt{3}}{4} \frac{F}{[(\sum w_a \gamma_a)(\sum w_a / \gamma_a)]^{\frac{1}{2}}}. \quad (33)$$

It is clear from equations (11) and (12) that, without loss of generality, we can impose on the γ_a 's a condition of the form

$$\sum_{a=1}^m w_a / \gamma_a = 1. \quad (34)$$

With such a "normalization," equation (33) becomes identical with the result first proved by Hopf.⁷

An explicit expression for the constants L_α occurring in our solution (21) can be obtained from equations (25) and (28); for, defining

$$R_\beta(\mu) = \prod_{\alpha \neq \beta} (1 - \mu k_\alpha) \quad (35)$$

and multiplying equation (25) by $R(\mu)$, we get

$$\sum_{\beta=1}^{nm-1} L_\beta R_\beta(\mu) + (Q - \mu)R(\mu) = (-1)^{nm} k_1 k_2 \dots k_{nm-1} P(\mu). \quad (36)$$

If we now set $\mu = 1/k_\beta$ in equation (36), we find

$$L_\beta = (-1)^{nm} k_1 k_2 \dots k_{nm-1} \frac{P(1/k_\beta)}{R_\beta(1/k_\beta)}. \quad (37)$$

The function $S(\mu)$ given by expression (28) also gives a closed expression for the emergent radiation in the different parts into which the spectrum has been divided. Thus, introducing equation (24) in the expression

$$I^{(a)}(0, \mu) = w_a \int_0^\infty B(t) e^{-t/\mu} \frac{dt}{\mu} \quad (38)$$

and making use of the definition (25), we obtain

$$I^{(a)}(0, \mu) = \frac{3}{4} F \frac{w_a}{\sum w_\alpha / \gamma_\alpha} S\left(-\frac{\mu}{\gamma_a}\right). \quad (39)$$

Thus we complete the description of the radiation field for the case of pure absorption.

3.3. *Numerical examples.*—We shall now give the numerical solutions in the first two approximations for the models considered in part B.

Example a: $w_1 = 0.8, \quad w_2 = 0.2, \quad \gamma = 5. \quad (40)$

In the first approximation the characteristic root is $k_1 = \sqrt{35} = 5.91608$, and the integrated Planck function is given by

$$B(\tau) = \frac{3}{4} \frac{F}{0.84} (\tau + 0.52379 - 0.12938 e^{-k_1 \tau}). \quad (41)$$

In the second approximation the characteristic roots are

$$k_1 = 1.91751, \quad k_2 = 5.00488, \quad k_3 = 11.98673, \quad (42)$$

and the expression for the source function is

$$B(\tau) = \frac{3}{4} \frac{F}{0.84} (\tau + 0.63660 - 0.13287 e^{-k_1 \tau} - 0.04505 e^{-k_2 \tau} - 0.06427 e^{-k_3 \tau}). \quad (43)$$

The corresponding relation between boundary and effective temperatures is

$$T_0 = 0.76034 T_e. \quad (44)$$

⁷ *Ibid.*, eq. (51).

The values of $B(\tau)$ given by expressions (41) and (43) are tabulated in Table 1 and further illustrated in Figure 1.

Example b: $w_1 = 0.8, \quad w_2 = 0.2, \quad \gamma = 10.$ (45)

First approximation:

$$k_1 = 9.37321, \quad (46)$$

$$B(\tau) = \frac{3}{4} \frac{F}{0.82} (\tau + 0.52840 - 0.21596 e^{-k_1 \tau}). \quad (47)$$

TABLE 1
THE INTEGRATED PLANCK FUNCTION $B(\tau)$ FOR THE VARIOUS MODELS

τ	$\gamma = \lambda = 5$		$\gamma = \lambda = 10$		$\gamma = 5, \lambda = 1$		$\gamma = 10, \lambda = 1$		$\gamma = 10, \lambda = 1$ $\gamma = \lambda = 1$	
	I*	II	I	II	I	II	I	II	I	II
0.00	0.3522	0.3522	0.2858	0.2858	0.4330	0.4097	0.4330	0.4308	0.4330	0.4362
0.02	0.3829	0.3905	0.3378	0.3526	0.4538	0.4348	0.4572	0.4603	0.4650	0.4724
0.05	0.4264	0.4424	0.4054	0.4335	0.4845	0.4712	0.4923	0.5019	0.5126	0.5236
0.10	0.4930	0.5180	0.4974	0.5379	0.5349	0.5295	0.5487	0.5664	0.5921	0.6045
0.20	0.6109	0.6462	0.6359	0.6942	0.6330	0.6400	0.6549	0.6856	0.7266	0.7364
0.30	0.7159	0.7590	0.7458	0.8203	0.7285	0.7454	0.7553	0.7972	0.8016	0.8209
0.40	0.8140	0.8645	0.8445	0.9328	0.8222	0.8475	0.8522	0.9039	0.8766	0.9037
0.50	0.9081	0.9659	0.9388	1.0380	0.9145	0.9472	0.9470	1.0072	0.9516	0.9851
0.60	1.0001	1.0645	1.0314	1.1391	1.0059	1.0450	1.0405	1.1079	1.0266	1.0654
0.80	1.1809	1.2564	1.2149	1.3343	1.1871	1.2364	1.2254	1.3041	1.1766	1.2233
1.00	1.3602	1.4435	1.3979	1.5246	1.3669	1.4238	1.4091	1.4956	1.3266	1.3786
1.40	1.7176	1.8103	1.7638	1.8981	1.7250	1.7910	1.7753	1.8709	1.6266	1.6846
1.60	1.8962	1.9914	1.9467	2.0829	1.9036	1.9726	1.9583	2.0564	1.7766	1.8362
2.00	2.2534	2.3515	2.3125	2.4508	2.2609	2.3331	2.3241	2.4251	2.0766	2.1380
3.00	3.1462	3.2466	3.2272	3.3668	3.1538	3.2287	3.2388	3.3418	2.8266	2.8893
4.00	4.0391	4.1398	4.1418	4.2815	4.0467	4.1220	4.1534	4.2568	3.5766	3.6395

* "I" and "II" refer to the first and second approximations, respectively.

Second approximation:

$$k_1 = 2.16772, \quad k_2 = 7.43277, \quad k_3 = 22.62302, \quad (48)$$

$$B(\tau) = \frac{3}{4} \frac{F}{0.82} (\tau + 0.68117 - 0.12370 e^{-k_1 \tau} - 0.18300 e^{-k_2 \tau} - 0.06205 e^{-k_3 \tau}). \quad (49)$$

The boundary value of the temperature in this example is

$$T_0 = 0.72114 T_e. \quad (50)$$

The numerical values of $B(\tau)$ determined by equations (47) and (49) are given in Table 1 and are graphically represented in Figure 1.

4.1. *Absorption lines formed in a process involving scattering as well as absorption.*—When $\epsilon \neq 1$, equations (7) and (8) can be written in the form

$$\left. \begin{aligned} \mu \frac{dI^{(1)}}{d\tau} &= I^{(1)} - w_1 B, \\ \mu \frac{dI^{(2)}}{d\tau} &= \gamma I^{(2)} - (\gamma - \lambda) J^{(2)} - w_2 \lambda B, \end{aligned} \right\} \quad (51)$$

where

$$\gamma = 1 + \xi \quad \text{and} \quad \lambda = 1 + \epsilon \xi. \quad (52)$$

The condition (9) for a constant net flux is

$$B = \frac{J^{(1)} + \lambda J^{(2)}}{w_1 + \lambda w_2} = \sigma (J^{(1)} + \lambda J^{(2)}), \quad (53)$$

and the system of equations to be solved becomes

$$\left. \begin{aligned} \mu \frac{dI^{(1)}}{d\tau} &= I^{(1)} - w_1 \sigma J^{(1)} - w_1 \lambda \sigma J^{(2)}, \\ \mu \frac{dI^{(2)}}{d\tau} &= \gamma I^{(2)} - (\gamma - w_1 \lambda \sigma) J^{(2)} - w_2 \lambda \sigma J^{(2)}. \end{aligned} \right\} \quad (54)$$

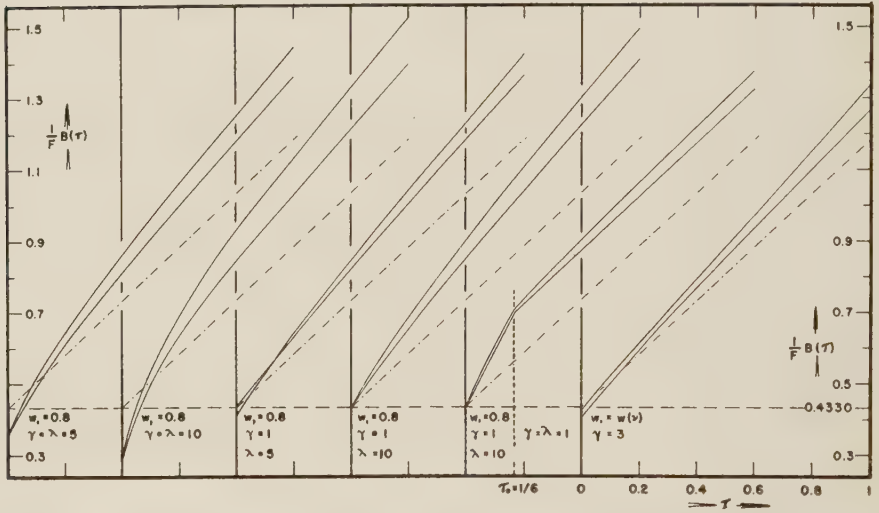


FIG. 1.—Comparison of the integrated Planck function in the various models for the first two approximations (solid lines) with the first approximation of the standard case (broken line). The scale of the abscissae is displaced by 0.4 units for each case considered. (The value of γ and λ in this diagram should be interchanged in every case but that on the extreme right.)

Again following the method developed in paper II, we find that the general solution of this system in the n th approximation is

$$\left. \begin{aligned} I_i^{(1)} &= \frac{3}{4} \frac{F w_1}{w_1 + w_2 / \gamma} \left[\tau + \mu_i + Q + \frac{1}{w_1} \sum_{a=1}^{2n-1} \left(\frac{L_a e^{-k_a \tau}}{1 + k_a \mu_i} + \frac{L_{-a} e^{k_a \tau}}{1 - k_a \mu_i} \right) \frac{1}{1 - G_a} \right], \\ I_i^{(2)} &= \frac{3}{4} \frac{F w_2}{w_1 + w_2 / \gamma} \left[\tau + \frac{\mu_i}{\gamma} + Q + \frac{1}{w_2 \gamma} \sum_{a=1}^{2n-1} \left(\frac{L_a e^{-k_a \tau}}{1 + k_a \mu_i / \gamma} \right. \right. \\ &\quad \left. \left. + \frac{L_{-a} e^{k_a \tau}}{1 - k_a \mu_i / \gamma} \right) \frac{1}{H_a - 1} \right] \quad (i = \pm 1, \pm 2, \dots, \pm n), \end{aligned} \right\} \quad (55)$$

where L_a , L_{-a} ($a = 1, 2, \dots, 2n - 1$), and Q are $4n - 1$ constants of integration, the k_a 's are the $2n - 1$ positive roots of the characteristic equation

$$1 - w_1 \sigma G - \left(1 - \frac{w_1 \lambda \sigma}{\gamma} \right) H + w_1 \sigma \left(1 - \frac{\lambda}{\gamma} \right) G H = 0, \quad (56)$$

and where, furthermore,

$$G = \frac{1}{2} \sum_{i=-n}^n \frac{a_i}{1 + k \mu_i} \quad \text{and} \quad H = \frac{1}{2} \sum_{i=-n}^n \frac{a_i}{1 + k \mu_i / \gamma}. \quad (57)$$

If the solution has to be asymptotically linear as $\tau \rightarrow \infty$, $L_{-a} = 0$ for every a , and equations (55) become

$$\left. \begin{aligned} I_i^{(1)} &= \frac{3}{4} \frac{F w_1}{w_1 + w_2 / \gamma} \left[\tau + \mu_i + Q + \frac{1}{w_1} \sum_{a=1}^{2n-1} \frac{L_a}{1 - G_a} \frac{e^{-k_a \tau}}{1 + k_a \mu_i} \right], \\ I_i^{(2)} &= \frac{3}{4} \frac{F w_2}{w_1 + w_2 / \gamma} \left[\tau + \frac{\mu_i}{\gamma} + Q + \frac{1}{w_2 \gamma} \sum_{a=1}^{2n-1} \frac{L_a}{H_a - 1} \frac{e^{-k_a \tau}}{1 + k_a \mu_i / \gamma} \right]. \end{aligned} \right\} \quad (58)$$

The $2n$ constants that appear in these equations are determined by the condition of vanishing incident radiation:

$$\left. \begin{aligned} \frac{1}{w_1} \sum_{a=1}^{2n-1} \frac{L_a}{1 - G_a} \frac{1}{1 - k_a \mu_i} + Q &= \mu_i, \\ \frac{1}{w_2 \gamma} \sum_{a=1}^{2n-1} \frac{L_a}{H_a - 1} \frac{1}{1 - k_a \mu_i / \gamma} + Q &= \frac{\mu_i}{\gamma} \end{aligned} \right\} \quad (i = 1, \dots, n). \quad (59)$$

The source function $B(\tau)$ in this case is given by

$$B(\tau) = \frac{3}{4} \frac{F}{w_1 + w_2 / \gamma} \left[\tau + Q + \sum_{a=1}^{2n-1} L_a e^{-k_a \tau} \left(\frac{G_a}{1 - G_a} + \frac{\lambda}{\gamma} \frac{H_a}{H_a - 1} \right) \sigma \right]. \quad (60)$$

It has not been possible to evaluate explicitly the value of $B(\tau)$ at $\tau = 0$. It is perhaps of interest, however, to mention that in the case of pure scattering ($\lambda = 1$), the first approximation always gives for the boundary temperature the exact value obtained in the standard case, independently of the values that w_1 , w_2 , and γ might have. However, this situation no longer holds in a higher approximation, because the value of $B(0)$ obtained then is smaller than the one given in the first approximation, as can be seen from the numerical examples given below.

4.2. Numerical examples.—Again we shall consider the two models given in paper B. Both refer to the extreme case in which the lines arise as a pure scattering phenomenon, i.e., $\epsilon = 0$ and $\lambda = 1$.

$$\text{Example a:} \quad w_1 = 0.8, \quad w_2 = 0.2, \quad \gamma = 5, \quad \lambda = 1. \quad (61)$$

$$\left. \begin{aligned} \text{First approximation:} \quad k_1 &= 3.54965, \quad Q = 0.53228, \\ G_1 &= -0.31250, \quad H_1 = 1.201923, \quad L_1 = -0.049666, \end{aligned} \right\} \quad (62)$$

$$B(\tau) = \frac{3}{4} \frac{F}{0.84} (\tau + 0.53228 - 0.04730 e^{-k_1 \tau}), \quad (63)$$

and

$$T_0 = 0.80119 T_e. \quad (64)$$

Second approximation:

$$\left. \begin{aligned} k_1 &= 1.81355, & k_2 &= 3.43621, & k_3 &= 10.55209, \\ G_1 &= 0.915925, & G_2 &= -1.83255, & G_3 &= -0.059204, \\ H_1 &= 1.053002, & H_2 &= 1.22516, & H_3 &= 1.14460, \\ L_1 &= -0.0089599, & L_2 &= -0.024963, & L_3 &= -0.0115666, \\ Q &= 0.61668, \end{aligned} \right\} \quad (65)$$

$$B(\tau) = \frac{3}{4} \frac{F}{0.84} (\tau + 0.61668 - 0.13321 e^{-k_1 \tau} - 0.01102 e^{-k_2 \tau} - 0.01359 e^{-k_3 \tau}), \quad (66)$$

and

$$T_0 = 0.80004 T_e. \quad (67)$$

Example b: $w_1 = 0.8, \quad w_2 = 0.2, \quad \gamma = 10, \quad \lambda = 1. \quad (68)$

First approximation: $k_1 = 4.95984, \quad Q = 0.54106,$

$$\left. \begin{aligned} G_1 &= -0.13889, & H_1 &= 1.08932, & L_1 &= -0.061620, \end{aligned} \right\} \quad (69)$$

$$B(\tau) = \frac{3}{4} \frac{F}{0.82} (\tau + 0.54106 - 0.06763 e^{-k_1 \tau}), \quad (70)$$

and

$$T_0 = 0.80119 T_e. \quad (71)$$

Second approximation:

$$\left. \begin{aligned} k_1 &= 1.94284, & k_2 &= 4.87106, & k_3 &= 20.38105, \\ G_1 &= 0.96358, & G_2 &= -0.39521, & G_3 &= -0.01500, \\ H_1 &= 1.01288, & H_2 &= 1.09266, & H_3 &= 1.08723, \\ L_1 &= -0.0037807, & L_2 &= -0.048638, & L_3 &= -0.0079513, \\ Q &= 0.65413, \end{aligned} \right\} \quad (72)$$

$$B(\tau) = \frac{3}{4} \frac{F}{0.82} (\tau + 0.65413 - 0.12976 e^{-k_1 \tau} - 0.04358 e^{-k_2 \tau} - 0.00979 e^{-k_3 \tau}), \quad (73)$$

and

$$T_0 = 0.80015 T_e. \quad (74)$$

The values of $B(\tau)$, determined on the two approximations of the two examples considered for different values of τ , are given in Table 1 and illustrated in Figure 1.

5.1. *The layer model.*—We shall consider now the case when the parameters that describe the radiation field are not constant throughout the atmosphere but have different values above and below a certain optical depth τ_0 (for continuous absorption). We shall suppose that in the higher layer ($\tau < \tau_0$) the process in which the lines arise is characterized by certain parameters (γ_a, λ_a), while in the bottom layer ($\tau > \tau_0$) these constants are (γ_b, λ_b). We shall further assume that w_2 is the same in both layers. According to the results of the preceding paragraph, the solution of the equations of transfer for $\tau < \tau_0$ will have the form (55); this involves $4n - 1$ constants of integration. Moreover, for $\tau > \tau_0$, the solution will be of the form (58), since it must be asymptotically linear as $\tau \rightarrow \infty$ and will therefore involve $2n$ further constants of integration. Consequently, we have a total of

$6n - 1$ constants of integration, which have to be determined from the boundary conditions. These conditions are the vanishing of the incident radiation at $\tau = 0$.

$$I_{-i}^{(1)}(\tau = 0) = 0, \quad I_{-i}^{(2)}(\tau = 0) = 0 \quad (i = 1, 2, \dots, n), \quad (75)$$

and the continuity of the solution at $\tau = \tau_0$,

$$I_i^{(1)}(\tau_0^-) = I_i^{(1)}(\tau_0^+), \quad I_i^{(2)}(\tau_0^-) = I_i^{(2)}(\tau_0^+) \quad (i = \pm 1, \dots, \pm n). \quad (76)$$

Of the $4n$ equations represented in relations (76), only $4n - 1$ are linearly independent, since our solution in the two regions satisfies the condition of constant net flux:

$$F = 2 \sum_{i=-n}^n a_i \mu_i (I_i^{(1)} + I_i^{(2)}) \tau_0^- = 2 \sum_{i=-n}^n a_i \mu_i (I_i^{(1)} + I_i^{(2)}) \tau_0^+. \quad (77)$$

Hence, the system of $6n$ linear equations, represented by equations (75) and (76), is of rank $6n - 1$ and determines uniquely the $6n - 1$ constants of integration which describe the solution of the problem. For the numerical solution of this system it is convenient to write down equations (76) in the alternative form

$$\left. \begin{aligned} \sum_{i=-n}^n a_i \mu_i^m I_i^{(1)}(\tau_0^-) &= \sum_{i=-n}^n a_i \mu_i^m I_i^{(1)}(\tau_0^+), \\ \sum_{i=-n}^n a_i \mu_i^m I_i^{(2)}(\tau_0^-) &= \sum_{i=-n}^n a_i \mu_i I_i^{(2)}(\tau_0^+) \end{aligned} \right\} \quad (m = 0, 1, \dots, 2n - 1). \quad (78)$$

For $m = 0$ these equations give

$$J^{(1)}(\tau_0^-) = J^{(1)}(\tau_0^+), \quad J^{(2)}(\tau_0^-) = J^{(2)}(\tau_0^+). \quad (79)$$

From the expression (53) for the integrated Planck function,

$$B(\tau) = \frac{J^{(1)} + \lambda J^{(2)}}{w_1 + \lambda w_2}, \quad (80)$$

we conclude that the continuity of $B(\tau)$ at τ_0 implies

$$\lambda_a = \lambda_b \quad \text{or} \quad (\epsilon \xi)_a = (\epsilon \xi)_b. \quad (81)$$

Thus we have proved that the temperature will be continuous at $\tau = \tau_0$ only when relation (81) is satisfied, a condition which was stated by Chandrasekhar in paper B for the Milne approximation.

A. Unsöld in his recent treatment of the subject⁸ has questioned the use made by Chandrasekhar in paper B of equation (81) as one of the conditions of the problem, on the grounds that the quantity $\epsilon \xi$ has nothing to do with the formulation of the problem of radiative equilibrium. Although this argument is formally correct, it should be pointed out that a model atmosphere with a discontinuity in its temperature distribution would not seem to have a physical basis.

5.2. *Layer model with pure scattering for $\tau < \tau_0$ and only continuous absorption for $\tau > \tau_0$.*—The discussion of a layer model for which $\lambda_a = 1$, $\gamma_a = \gamma$, and $\lambda_b = \gamma_b = \eta$ is of interest, since in our picture of the problem for $\eta = 1$ we approach the original "blanketing model" first considered by Milne⁹ and Lindblad.¹⁰ According to the general method outlined in the preceding paragraph, the solution for $\tau < \tau_0$ will have the form

⁸ *Zs. f. Ap.*, **22**, 356, 1943.

⁹ *Op. cit.*, p. 225.

¹⁰ *Nova Acta Reg. Soc. Sci., Upsala*, **6**, No. 1, 17, 1923.

$$\left. \begin{aligned} I_i^{(1)} &= \frac{3}{4} \frac{Fw_1}{w_1 + w_2/\gamma} \left\{ \tau + \mu_i + Q + \frac{1}{w_1} \sum_{a=1}^{2n-1} \left(\frac{L_a e^{-k_a \tau}}{1 + \mu_i k_a} + \frac{L_{-a} e^{k_a \tau}}{1 - \mu_i k_a} \right) \frac{1}{1 - G_a} \right\}, \\ I_i^{(2)} &= \frac{3}{4} \frac{Fw_2}{w_1 + w_2/\gamma} \left\{ \tau + \frac{\mu_i}{\gamma} + Q + \frac{1}{w_2 \gamma} \sum_{a=1}^{2n-1} \left(\frac{L_a e^{-k_a \tau}}{1 + \mu_i k_a/\gamma} \right. \right. \\ &\quad \left. \left. + \frac{L_{-a} e^{k_a \tau}}{1 - \mu_i k_a/\gamma} \right) \frac{1}{H_a - 1} \right\} \quad (i = \pm 1, \dots, \pm n), \end{aligned} \right\} \quad (82)$$

and for $\tau > \tau_0$

$$\left. \begin{aligned} I_i^{(1)} &= \frac{3}{4} \frac{Fw_1}{w_1 + w_2/\gamma} \left\{ \tau + \mu_i + R + \sum_{a=1}^{2n-1} \frac{M_a e^{-q_a \tau}}{1 + \mu_i q_a} \right\}, \\ I_i^{(2)} &= \frac{3}{4} \frac{Fw_2}{w_1 + w_2/\gamma} \left\{ \tau + \frac{\mu_i}{\gamma} + R + \sum_{a=1}^{2n-1} \frac{M_a e^{-q_a \tau}}{1 + \mu_i q_a/\gamma} \right\}. \end{aligned} \right\} \quad (83)$$

The k_a 's appearing in equation (82) are the roots of equation (56), while the q_a 's occurring in equation (83) are the roots of the equation

$$w_1 \sum_{i=1}^n \frac{a_i}{1 - \mu_i^2 q^2} + w_2 \eta \sum_{i=1}^n \frac{a_i}{1 - \mu_i^2 q^2/\eta^2} = w_1 + w_2 \eta. \quad (84)$$

The boundary conditions (75) and (78), which determine the $6n - 1$ constants of integration, are readily written down and solved numerically without difficulty. However, if we let $\eta \rightarrow 1$, the solution of these equations is not straightforward, and some care must be taken in the limiting process. If we introduce the definitions

$$D_{m,a} = \sum_{i=1}^n \frac{a_i \mu_i^m}{1 - \mu_i^2 q_a^2}, \quad E_{m,a} = \sum_{i=1}^n \frac{a_i \mu_i^m}{1 - \mu_i^2 q_a^2/\eta^2}, \quad (85)$$

the boundary conditions (78), with their right-hand sides in explicit form, can be written in the following fashion:

$$\left. \begin{aligned} \sum_{i=-n}^n a_i \mu_i^{2j} I_i^{(1)}(\tau_0^-) &= \frac{3}{2} \frac{Fw_1}{w_1 + w_2/\eta} \left\{ \frac{\tau_0 + R}{2j+1} + \sum_{a=1}^{2n-1} M_a e^{-q_a \tau_0} D_{2j,a} \right\}, \\ \sum_{i=-n}^n a_i \mu_i^{2j} I_i^{(2)}(\tau_0^-) &= \frac{3}{2} \frac{Fw_2}{w_1 + w_2/\eta} \left\{ \frac{\tau_0 + R}{2j+1} + \sum_{a=1}^{2n-1} M_a e^{-q_a \tau_0} E_{2j,a} \right\}, \\ \sum_{i=-n}^n a_i \mu_i^{2j+1} I_i^{(1)}(\tau_0^-) &= \frac{3}{2} \frac{Fw_1}{w_1 + w_2/\eta} \left\{ \frac{1}{2j+3} + \sum_{a=1}^{2n-1} M_a e^{-q_a \tau_0} D_{2j+1,a} \right\}, \\ \sum_{i=-n}^n a_i \mu_i^{2j+1} I_i^{(2)}(\tau_0^-) &= \frac{3}{2} \frac{Fw_2}{w_1 + w_2/\eta} \left\{ \frac{1/\eta}{2j+3} + \sum_{a=1}^{2n-1} M_a e^{-q_a \tau_0} E_{2j+1,a} \right\}. \end{aligned} \right\} \quad (86)$$

If we now write the characteristic equation (84) in the form of a polynomial in q^2 and then let $\eta \rightarrow 1$, we find that $n - 1$ of its roots tend to the characteristic roots of the "standard case," i.e., the positive roots of the equation

$$\sum_{i=1}^n \frac{a_i}{1 - \mu_i^2 q^2} = 1, \quad (87)$$

while the other n roots tend to the reciprocals $1/\mu_i$ of the n -positive zeros of the $2n$ th Legendre polynomial. On the other hand, from definitions (85) it is a simple matter¹¹ to obtain the following recursion formula:

$$\left. \begin{aligned} D_{2j-1,a} &= \frac{1}{(2j-1)q_a} + \frac{1}{(2j-3)q_a^3} + \dots + \frac{1-D_{0,a}}{q_a^{2j-1}}, \\ D_{2j,a} &= -\frac{1}{(2j-1)q_a^2} - \frac{1}{(2j-3)q_a^4} - \dots - \frac{1-D_{0,a}}{q_a^{2j}}, \\ E_{2j-1,a} &= \frac{\eta}{(2j-1)q_a} + \frac{\eta^3}{(2j-3)q_a^3} + \dots + \frac{1-E_{0,a}}{q_a^{2j-1}}\eta^{2j-1}, \\ E_{2j,a} &= -\frac{\eta^2}{(2j-1)q_a^2} - \frac{\eta^4}{(2j-3)q_a^4} - \dots - \frac{1-E_{0,a}}{q_a^{2j}}\eta^{2j} \end{aligned} \right\} \quad (88)$$

$(j = 1, 2, \dots, n-1).$

Making use of these expressions and letting $\eta \rightarrow 1$ in equations (86), we see that for the q_a 's that tend to $1/\mu_i$, the corresponding values of $D_{0,i}$ and $E_{0,i}$ will diverge to infinity, and equations (86) can be satisfied only if the corresponding values of M_i tend to zero in such a way that the limits,

$$\lim_{\eta \rightarrow 1} M_i D_{0,i} = \frac{\mathfrak{L}_i}{w_1} \quad \text{and} \quad \lim_{\eta \rightarrow 1} M_i E_{0,i} = \frac{\mathfrak{L}'_i}{w_2}, \quad (89)$$

are finite for every i . Furthermore, the condition of constant net flux requires that these limits $\mathfrak{L}_i$ and $\mathfrak{L}'_i$ satisfy the relation

$$\frac{\mathfrak{L}_i}{\mathfrak{L}'_i} = -1 \quad (i = 1, 2, \dots, n). \quad (90)$$

As a result of this limiting process, we can write the complete boundary conditions in the form

$$\left. \begin{aligned} I_{-\frac{1}{2}}^{(1)}(\tau = 0) &= 0, \\ I_{-\frac{1}{2}}^{(2)}(\tau = 0) &= 0 \end{aligned} \right\} \quad (i = 1, 2, \dots, n), \quad (91)$$

and

$$\left. \begin{aligned} \sum_{i=-n}^n a_i \mu_i^{2j} I_i^{(1)}(\tau_0^-) &= \frac{3}{2} F w_1 \left\{ \frac{\tau_0 + R}{2j+1} + \sum_{a=1}^{n-1} M_a D_{2j,a} e^{-q_a \tau_0} + \frac{1}{w_1} \sum_{i=1}^n \mu_i^{2j} \mathfrak{L}_i \right\}, \\ \sum_{i=-n}^n a_i \mu_i^{2j} I_i^{(2)}(\tau_0^-) &= \frac{3}{2} F w_2 \left\{ \frac{\tau_0 + R}{2j+1} + \sum_{a=1}^{n-1} M_a D_{2j,a} e^{-q_a \tau_0} - \frac{1}{w_2} \sum_{i=1}^n \mu_i^{2j} \mathfrak{L}'_i \right\}, \\ \sum_{i=-n}^n a_i \mu_i^{2j+1} I_i^{(1)}(\tau_0^-) &= \frac{3}{2} F w_1 \left\{ \frac{1}{2j+3} + \sum_{a=1}^{n-1} M_a D_{2j+1,a} e^{-q_a \tau_0} - \frac{1}{w_1} \sum_{i=1}^n \mu_i^{2j+1} \mathfrak{L}_i \right\}, \\ \sum_{i=-n}^n a_i \mu_i^{2j+1} I_i^{(2)}(\tau_0^-) &= \frac{3}{2} F w_2 \left\{ \frac{1}{2j+3} + \sum_{a=1}^{n-1} M_a D_{2j+1,a} e^{-q_a \tau_0} + \frac{1}{w_2} \sum_{i=1}^n \mu_i^{2j+1} \mathfrak{L}'_i \right\} \end{aligned} \right\} \quad (92)$$

$(j = 0, 1, \dots, n-1).$

¹¹ Cf. Paper VII, pp. 335-336.

If we introduce the values of $I_i^{(1)}$ and $I_i^{(2)}$ given by the relations (82) in the left-hand sides of equations (91) and (92), the determination of the constants is immediate.

The source function, $B(\tau)$, then takes the form

$$\left. \begin{aligned} B(\tau) &= \frac{3}{4} \frac{F}{w_1 + w_2/\gamma} \left\{ \tau + Q + \sum_{a=1}^{2n-1} c_a (L_a e^{-k_a \tau} + L_{-a} e^{k_a \tau}) \right\} & (\tau \leq \tau_0), \\ B(\tau) &= \frac{3}{4} F \left\{ \tau + R + \sum_{a=1}^{n-1} M_a e^{-q_a \tau} \right\} & (\tau \geq \tau_0), \end{aligned} \right\} \quad (93)$$

where

$$c_a = \frac{G_a}{1 - G_a} + \frac{1}{\gamma} \frac{H_a}{H_a - 1}. \quad (94)$$

If equations (93) are introduced into the expression for the emergent intensity in the continuum,

$$I^{(1)}(0, \mu) = w_1 \int_0^\infty B(\tau) e^{-\tau/\mu} \frac{d\tau}{\mu}, \quad (95)$$

we find that

$$\left. \begin{aligned} I^{(1)}(0, \mu) &= \frac{3}{4} F w_1 \left\{ \frac{1}{w_1 + w_2/\gamma} \left[\mu + Q - (\tau_0 + \mu + Q) e^{-\tau_0/\mu} \right. \right. \\ &\quad \left. \left. - \sum_{a=1}^{2n-1} c_a \left(L_a \frac{e^{-(1+k_a \mu)\tau_0/\mu} - 1}{1 + k_a \mu} + L_{-a} \frac{e^{-(1-k_a \mu)\tau_0/\mu} - 1}{1 - k_a \mu} \right) \right] \right. \\ &\quad \left. + e^{-\tau_0/\mu} \left[\tau_0 + \mu + R + \sum_{a=1}^{n-1} M_a \frac{e^{-\tau_0 q_a}}{1 + q_a \mu} \right] \right\}. \end{aligned} \right\} \quad (96)$$

The expression for the emergent intensity in the lines $I^{(2)}(0, \mu)$ is obtained if in equation (96) we substitute, throughout, μ/γ for μ and change the factor w_1 in front of the bracket by w_2 .

The expression for the total emergent intensity, $I(0, \mu)$, is then given by

$$I(0, \mu) = I^{(1)}(0, \mu) + I^{(2)}(0, \mu), \quad (97)$$

and the resulting law of darkening is

$$\phi(\mu) = \frac{I(0, \mu)}{I(0, 1)}. \quad (98)$$

5.3. *Numerical example.*—We have computed the example

$$w_1 = 0.8, \quad w_2 = 0.2, \quad \tau_0 = \frac{1}{6}, \quad \gamma = 10, \quad \lambda = 1, \quad \eta = 1, \quad (99)$$

considered in paper B.

The characteristic roots for $\tau < \tau_0$ have been computed already in example *b* of section 4.2, while the characteristic roots for $\tau > \tau_0$ are taken from paper II.

First approximation:

$$\left. \begin{aligned} k_1 &= 4.95984, & G_1 &= -0.13889, & H_1 &= 1.0893246, \\ L_1 &= -0.086740, & L_{-1} &= 0.052080, & Q &= 0.51147, \\ R &= 0.76886, & \mathfrak{L}_1 &= -0.131976, \end{aligned} \right\} \quad (100)$$

$$\left. \begin{aligned} B(\tau) &= \frac{3}{4} \frac{F}{0.82} (\tau + 0.51147 - 0.09520 e^{-k_1 \tau} + 0.05716 e^{k_1 \tau}) & (\tau \leq \tau_0), \\ B(\tau) &= \frac{3}{4} F (\tau + 0.76886) & (\tau \geq \tau_0). \end{aligned} \right\} \quad (101)$$

The dependence on τ of $B(\tau)$ is given in Table 1 and illustrated in Figure 1. We may notice that the value of $B(\tau)$ at $\tau = 0$ is now the same as that in the standard model,

$$B(0) = 0.43301 F. \quad (102)$$

Second approximation:

$$\left. \begin{aligned} k_1 &= 1.94285, & k_2 &= 4.87104, & k_3 &= 20.38105, \\ G_1 &= 0.9635599, & G_2 &= -0.3952121, & G_3 &= -0.0150044, \\ H_1 &= 1.0128751, & H_2 &= 1.0926626, & H_3 &= 1.0872368, \\ L_1 &= -0.0039855, & L_2 &= -0.064554, & L_3 &= -0.010949, \\ L_{-1} &= 0.0009266, & L_{-2} &= 0.036146, & L_{-3} &= -0.0001520, \\ M_1 &= -0.10505, & \mathfrak{L}_1 &= -0.062414, & \mathfrak{L}_2 &= -0.0022012, \\ q_1 &= 1.97203, & R &= 0.85271, & Q &= 0.62098. \end{aligned} \right\} \quad (103)$$

The values of $B(\tau)$ obtained with these constants are given in Table 1 and are plotted in Figure 1. We have computed for this model the darkening in the continuum and in the lines, as well as the total darkening. The resulting values are given in Table 2 for

TABLE 2
LIMB DARKENING IN THE LAYER MODEL

μ	$\frac{1}{F} I^{(1)}(0, \mu)$		$\frac{1}{F} I^{(2)}(0, \mu)$		$\frac{1}{F} I(0, \mu)$		$\phi(\mu)$	
	I*	II	I	II	I	II	I	II
0.0.....	0.3464	0.3490	0.0866	0.0872	0.4330	0.4362	0.3749	0.3633
0.1.....	0.4617	0.4702	.0898	.0909	0.5515	0.5611	0.4774	0.4673
0.2.....	0.5436	0.5553	.0930	.0943	0.6365	0.6497	0.5511	0.5411
0.3.....	0.6142	0.6298	.0962	.0977	0.7104	0.7274	0.6059	0.6150
0.4.....	0.6801	0.6990	.0994	.1009	0.7794	0.8000	0.6748	0.6663
0.5.....	0.7439	0.7654	.1023	.1041	0.8461	0.8695	0.7354	0.7245
0.6.....	0.8064	0.8308	.1051	.1067	0.9116	0.9375	0.7892	0.7809
0.7.....	0.8684	0.8948	.1079	.1098	0.9763	1.0046	0.8453	0.8376
0.8.....	0.9300	0.9583	.1105	.1125	1.0405	1.0708	0.9008	0.8919
0.9.....	0.9911	1.0207	.1130	.1151	1.1041	1.1358	0.9559	0.9460
1.0.....	1.0397	1.0830	0.1154	0.1176	1.1551	1.2006	1.0000	1.0000

* "I" and "II" refer to the first and the second approximations, respectively.

the first two approximations. The results for the second approximation are shown in Figure 2, where we have also exhibited the values of the observed limb darkening for integrated light in the sun, as reduced by Minnaert,¹² using Abbot's measures of the monochromatic darkening and Mulders' intensity distribution with wave length at the center of the solar disk. The agreement between the observed values and the computed total darkening is seen to be very satisfactory down to a distance of 0.05 from the limb, in units of the solar radius. The significance of this agreement will be considered in the last section of this article, in connection with the general discussion of the phenomenon of darkening.

6.1. *Absorption lines not uniformly distributed throughout the spectrum.*—In the preceding sections we have assumed that the distribution function of the lines is independent of frequency. We shall now construct an iteration method for solving the equations of radiative transfer in an atmosphere in which the lines arise in pure local thermodynamical

¹² *Zs. f. Ap.*, **13**, 196, 1937.

equilibrium, on the assumption that the probability distribution of the lines along the spectrum is a given function of frequency. The hypothesis that the lines arise in pure absorption is introduced not only for the sake of simplicity in the calculations, but also because, as Unsöld has pointed out,¹³ the true process in which the majority of non-resonance lines arise must have, to a great extent, the character of an absorption process. Accordingly, we shall write the equations of transfer (4) and (5) in the form

$$\left. \begin{aligned} \mu \frac{dI^{(1)}}{d\tau} &= I^{(1)} - \int_0^\infty w_1(\nu) B_\nu d\nu, \\ \frac{\mu}{\gamma} \frac{dI^{(2)}}{d\tau} &= I^{(2)} - \int_0^\infty w_2(\nu) B_\nu d\nu, \end{aligned} \right\} \quad (104)$$

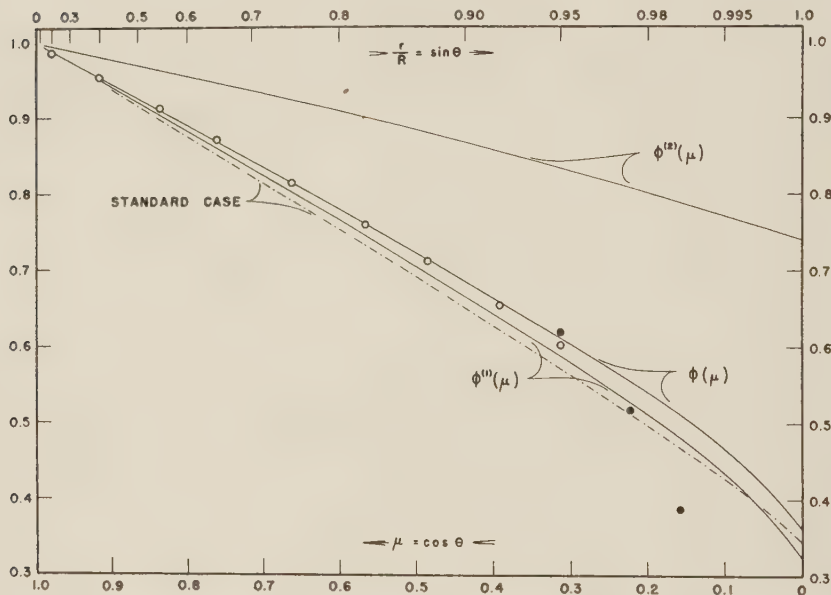


FIG. 2.—Comparison between the total darkening in the layer model and the observed integrated darkening in the solar disk. Open circles are points obtained from Abbot's observations of monochromatic darkening, and filled circles represent values reduced from measurements by Moll, Burger, and van der Bilt.

where $\gamma = 1 + \xi$ is supposed to be independent of depth and frequency. If we now define the averages

$$\left. \begin{aligned} \bar{w}_1(\tau) &= \frac{1}{B(T_\tau)} \int_0^\infty w_1(\nu) B_\nu(T_\tau) d\nu, \\ \bar{w}_2(\tau) &= \frac{1}{B(T_\tau)} \int_0^\infty w_2(\nu) B_\nu(T_\tau) d\nu, \end{aligned} \right\} \quad (105)$$

we can write

$$\bar{w}_1(\tau) = \omega_1 - \delta(\tau) \quad \text{and} \quad w_2(\tau) = \omega_2 + \delta(\tau), \quad (106)$$

with

$$\omega_1 + \omega_2 = 1. \quad (107)$$

¹³ Cf. *op. cit.*, p. 369.

In equation (106) ω_1 is a constant to be chosen appropriately. We can now write equations (104) in the form

$$\left. \begin{aligned} \mu \frac{dI^{(1)}}{d\tau} &= I^{(1)} - \omega_1 B(\tau) + \delta(\tau) B(\tau), \\ \frac{\mu}{\gamma} \frac{dI^{(2)}}{d\tau} &= I^{(2)} - \omega_2 B(\tau) - \delta(\tau) B(\tau). \end{aligned} \right\} \quad (108)$$

These equations have to be solved under the condition of constant net flux, expressed by the relation

$$J^{(1)} + \gamma J^{(2)} = (\omega_1 + \gamma \omega_2) B(\tau) + (\gamma - 1) \delta(\tau) B(\tau). \quad (109)$$

The iteration process that will be devised for the solution of equations (108) is entirely similar to the methods used by Chandrasekhar in paper VII and by Miss Tuberg¹⁴ in her discussion of the formation of absorption lines. The fundamental idea of the process is the following: If the solution of the equations

$$\left. \begin{aligned} \mu \frac{dI^{(1)}}{d\tau} &= I^{(1)} - \omega_1 B, \\ \frac{\mu}{\gamma} \frac{dI^{(2)}}{d\tau} &= I^{(2)} - \omega_2 B, \end{aligned} \right\} \quad (110)$$

under the condition

$$J^{(1)} + \gamma J^{(2)} = (\omega_1 + \gamma \omega_2) B, \quad (111)$$

gives a first approximation $B_1(\tau)$ to the source function $B(\tau)$, then the second approximation $B_2(\tau)$ will be given by the solution of the system

$$\left. \begin{aligned} \mu \frac{dI^{(1)}}{d\tau} &= I^{(1)} - \omega_1 B_2(\tau) + \delta(\tau) B_1(\tau), \\ \frac{\mu}{\gamma} \frac{dI^{(2)}}{d\tau} &= I^{(2)} - \omega_2 B_2(\tau) - \delta(\tau) B_1(\tau), \end{aligned} \right\} \quad (112)$$

with

$$J^{(1)} + \gamma J^{(2)} - (\gamma - 1) \delta(\tau) B_1(\tau) = (\omega_1 + \gamma \omega_2) B_2(\tau). \quad (113)$$

However, in order to compute the integrals appearing in the averages (105), we must have a knowledge of a zero-order approximation for the Planck function. We shall suppose that such a zero-order approximation is given by the distribution of temperature in the gray atmosphere. This assumption is justified by the fact that in practical cases the small uncertainty arising from the use of an approximate temperature distribution is negligible when compared with the errors caused by the uncertainty with which we can define numerically the function $w_2(\nu)$.

The formal generalizations that are necessary to define the n th iteration are easily written down; but we do not wish to go into these refinements of the theory, since we shall restrict our numerical computations to the first iteration, defined by equations (110)–(113).

The solution of equations (110), as we have seen, is

$$\left. \begin{aligned} I_i^{(1)} &= \frac{3}{4} \frac{F\omega_1}{\omega_1 + \omega_2/\gamma} \left\{ \tau + \mu_i + Q + \sum_{a=1}^{2n-1} \frac{L_a e^{-k_a \tau}}{1 + \mu_i k_a} \right\}, \\ I_i^{(2)} &= \frac{3}{4} \frac{F\omega_2}{\omega_1 + \omega_2/\gamma} \left\{ \tau + \frac{\mu_i}{\gamma} + Q + \sum_{a=1}^{2n-1} \frac{L_a e^{-k_a \tau}}{1 + \mu_i k_a/\gamma} \right\} \end{aligned} \right\} \quad (114)$$

$$(i = \pm 1, \dots, \pm n),$$

¹⁴ *Ap. J.*, **103**, 145, 1946.

where the k_a 's are the $2n - 1$ positive roots of the equation

$$\omega_1 + \omega_2 \gamma = \omega_1 \sum_{i=1}^n \frac{a_i}{1 - \mu_i^2 k^2} + \omega_2 \gamma \sum_{i=1}^n \frac{a_i}{1 - \mu_i^2 k^2 / \gamma^2} \quad (115)$$

and the L_a 's and Q are given by the boundary conditions. The corresponding integrated Planck function is

$$B_1(\tau) = \frac{3}{4} \frac{F}{w_1 + w_2 / \gamma} \left(\tau + Q + \sum_{a=1}^{2n-1} L_a e^{-k_a \tau} \right). \quad (116)$$

The system that we have to solve is obtained by combining equations (112) and (113) in the form

$$\left. \begin{aligned} \mu \frac{dI^{(1)}}{d\tau} &= I^{(1)} - \omega_1 \frac{J^{(1)} + \gamma J^{(2)}}{\omega_1 + \gamma \omega_2} + \gamma \frac{\delta(\tau) B_1(\tau)}{\omega_1 + \gamma \omega_2}, \\ \mu \frac{dI^{(2)}}{\gamma d\tau} &= I^{(2)} - \omega_2 \frac{J^{(1)} + \gamma J^{(2)}}{\omega_1 + \gamma \omega_2} - \gamma \frac{\delta(\tau) B_1(\tau)}{\omega_1 + \gamma \omega_2}. \end{aligned} \right\} \quad (117)$$

Since the homogeneous part of these equations has the form of system (110), we seek for equations (118) a solution of the form

$$\left. \begin{aligned} I_i^{(1)} &= \frac{3}{4} \frac{F \omega_1}{\omega_1 + \omega_2 / \gamma} \left\{ \tau + \mu_i + \mathcal{Q}(\tau) + \sum_{a=1}^{2n-1} \left[\frac{\mathcal{L}_a(\tau) e^{-k_a \tau}}{1 + \mu_i k_a} + \frac{\mathcal{L}_{-a}(\tau) e^{k_a \tau}}{1 - \mu_i k_a} \right] \right\}, \\ I_i^{(2)} &= \frac{3}{4} \frac{F \omega_2}{\omega_1 + \omega_2 / \gamma} \left\{ \tau + \frac{\mu_i}{\gamma} + \mathcal{Q}(\tau) + \sum_{a=1}^{2n-1} \left[\frac{\mathcal{L}_a(\tau) e^{-k_a \tau}}{1 + \mu_i k_a / \gamma} + \frac{\mathcal{L}_{-a}(\tau) e^{k_a \tau}}{1 - \mu_i k_a / \gamma} \right] \right\} \end{aligned} \right\} \quad (118)$$

$(i = \pm 1, \pm 2, \dots, \pm n).$

Substituting these equations in system (117), written for the points μ_i , we readily find that the variational equations for the determination of the $4n - 1$ functions $\mathcal{L}_a(\tau)$, $\mathcal{L}_{-a}(\tau)$, and $\mathcal{Q}(\tau)$ are:

$$\left. \begin{aligned} \mu_i \left\{ \frac{d\mathcal{Q}}{d\tau} + \sum_{a=1}^{2n-1} \left[\frac{e^{-k_a \tau}}{1 + \mu_i k_a} \frac{d\mathcal{L}_a}{d\tau} + \frac{e^{k_a \tau}}{1 - \mu_i k_a} \frac{d\mathcal{L}_{-a}}{d\tau} \right] \right\} &= \frac{\Delta(\tau)}{w_1}, \\ \mu_i \left\{ \frac{d\mathcal{Q}}{d\tau} + \sum_{a=1}^{2n-1} \left[\frac{e^{-k_a \tau}}{1 + \mu_i k_a / \gamma} \frac{d\mathcal{L}_a}{d\tau} + \frac{e^{k_a \tau}}{1 - \mu_i k_a / \gamma} \frac{d\mathcal{L}_{-a}}{d\tau} \right] \right\} &= -\frac{\Delta(\tau)}{w_2}, \end{aligned} \right\} \quad (119)$$

where

$$\Delta(\tau) = \frac{\omega_1 + \omega_2 / \gamma}{\omega_1 + \gamma \omega_2} \frac{4}{3F} \gamma \delta(\tau) B_1(\tau). \quad (120)$$

It is possible to write down explicitly the solution of the $2n$ variational equations (119) by following a procedure entirely similar to that followed by Chandrasekhar in paper VII. But again, since we shall restrict our numerical calculations to the case $n = 1$, we do not wish to go into these developments. For the case $n = 1$, equations (119) give immediately

$$\frac{d\mathcal{Q}}{d\tau} = 0, \quad (121)$$

$$e^{-k\tau} \frac{d\mathcal{L}_1}{d\tau} = \frac{k}{2} \frac{1 - 1/\gamma^2}{w_1 + w_2 / \gamma} \Delta(\tau), \quad (122)$$

and

$$e^{k\tau} \frac{d\mathcal{L}_{-1}}{d\tau} = -\frac{k}{2} \frac{1 - 1/\gamma^2}{w_1 + w_2/\gamma} \Delta(\tau), \quad (123)$$

from which the solution for $\mathcal{Q}$, $\mathcal{L}_1(\tau)$, and $\mathcal{L}_{-1}(\tau)$ follows. We have

$$\mathcal{Q} = \text{constant} = Q + \Delta Q, \quad (124)$$

$$\mathcal{L}_1(\tau) = \frac{k}{2} \frac{1 - 1/\gamma^2}{w_1 + w_2/\gamma} \int_0^\tau e^{k\tau} \Delta(\tau) d\tau + L_1 + \Delta L_1, \quad (125)$$

and

$$\mathcal{L}_{-1}(\tau) = \frac{k}{2} \frac{1 - 1/\gamma^2}{w_1 + w_2/\gamma} \int_\tau^\infty e^{-k\tau} \Delta(\tau) d\tau, \quad (126)$$

where ΔQ and ΔL_1 are constants of integration. It is to be noticed, however, that, in integrating equation (123) in the form (126), the constant of integration has been so chosen as to satisfy the requirement that the values of $I_i^{(1)}$ and $I_i^{(2)}$ given by equations (118) do not increase exponentially as $\tau \rightarrow \infty$.

Finally, the constants of integration, ΔL_1 and ΔQ , are to be determined from the condition of vanishing incident radiation,

$$\left. \begin{aligned} \frac{L_1 + \Delta L_1}{1 - \mu_1 k} + \frac{\mathcal{L}_{-1}(0)}{1 + \mu_1 k} + Q + \Delta Q &= \mu_1, \\ \frac{L_1 + \Delta L_1}{1 - \mu_1 k/\gamma} + \frac{\mathcal{L}_{-1}(0)}{1 + \mu_1 k/\gamma} + Q + \Delta Q &= \frac{\mu_i}{\gamma}, \end{aligned} \right\} \quad (127)$$

or, since L_1 and Q are so chosen as to satisfy

$$\left. \begin{aligned} \frac{L_1}{1 - \mu_1 k} + Q &= \mu_1, \\ \frac{L_1}{1 - \mu_1 k/\gamma} + Q &= \frac{\mu_1}{\gamma}, \end{aligned} \right\} \quad (128)$$

we are left with

$$\left. \begin{aligned} \frac{\Delta L_1}{1 - \mu_1 k} + \Delta Q &= -\frac{\mathcal{L}_{-1}(0)}{1 + \mu_1 k}, \\ \frac{\Delta L_1}{1 - \mu_1 k/\gamma} + \Delta Q &= -\frac{\mathcal{L}_{-1}(0)}{1 + \mu_1 k/\gamma}, \end{aligned} \right\} \quad (129)$$

for the determination of ΔQ and ΔL_1 .

On the other hand, since the constant ω_2 has been left unspecified up to the present, we now determine it by the condition that $\mathcal{L}_{-1}(0)$ vanish, i.e.,

$$\int_0^\infty e^{-k\tau} \Delta(\tau) d\tau = 0. \quad (130)$$

Equations (106) and (120), together with equation (130), give

$$\omega_2 = \frac{\int_0^\infty e^{-k\tau} B_1(\tau) \bar{w}_2(\tau) d\tau}{\int_0^\infty e^{-k\tau} B_1(\tau) d\tau}. \quad (131)$$

If in this equation we express the integrals by numerical quadrature formulas, in terms of the zeros of the Laguerre polynomials and the associated Christoffel numbers λ_i , we obtain

$$\omega_2 = \frac{\sum_{i=1}^n \lambda_i B_1(\tau_i/k) \bar{w}_2(\tau_i/k)}{\sum_{i=1}^n \lambda_i B_1(\tau_i/k)}. \quad (132)$$

It is seen that ω_2 is a weighted average of $\bar{w}_2(\tau)$. However, it should be noticed that the weights depend on ω_2 and that, therefore, the value of ω_2 has to be found by a method of trial and error.

When equation (130) is satisfied, the integration constants ΔL_1 and ΔQ , according to equation (129), have the values

$$\Delta Q = 0 \quad \text{and} \quad \Delta L_1 = 0; \quad (133)$$

and our solution (118) takes the form

$$\left. \begin{aligned} I_i^{(1)} &= \frac{3}{4} \frac{F\omega_1}{\omega_1 + \omega_2/\gamma} \left(\tau + \mu_i + Q + \frac{\mathcal{L}_1(\tau) e^{-k\tau}}{1 + \mu_i k} + \frac{\mathcal{L}_{-1}(\tau) e^{k\tau}}{1 - \mu_i k} \right), \\ I_i^{(2)} &= \frac{3}{4} \frac{F\omega_2}{\omega_1 + \omega_2/\gamma} \left(\tau + \frac{\mu_i}{\gamma} + Q + \frac{\mathcal{L}_1(\tau) e^{-k\tau}}{1 + \mu_i k/\gamma} + \frac{\mathcal{L}_{-1}(\tau) e^{k\tau}}{1 - \mu_i k/\gamma} \right) \end{aligned} \right\} \quad (134)$$

($i = \pm 1$).

Finally, the integrated Planck function takes the form

$$B(\tau) = B_1(\tau) \left[1 - \frac{\gamma - 1}{\omega_1 + \gamma\omega_2} \delta(\tau) \right] + \frac{3}{4} F \frac{\Psi(\tau)}{\omega_1 + \omega_2/\gamma}, \quad (135)$$

where

$$\Psi(\tau) = \frac{k}{2} \frac{1 - 1/\gamma^2}{\omega_1 + \omega_2/\gamma} \int_0^\infty e^{-k|t-\tau|} \Delta(t) dt. \quad (136)$$

It may be further noticed that, as a consequence of the procedure followed to determine the constants of integration, the emergent fluxes $F^{(1)}$ and $F^{(2)}$ in this (2, 1) approximation retain the values given by the (1, 1) approximation.

6.2. *Application to the solar atmosphere.*—In order to apply the theory developed in the preceding paragraph to the case of the sun, we need, first, to define numerically the distribution function of the lines $w_2(\nu)$. We shall take as a measure of the probability distribution of the lines the distribution with frequency of the total absorption $A(\nu)$ produced by the Fraunhofer lines in the solar spectrum. The dependence on frequency of this total absorption, as determined by Mulders,¹⁵ is shown graphically in Figure 3, together with the smooth curve adopted for $A(\nu)$. It should be pointed out that, although his values are uncertain below λ 3900 Å, this uncertainty is not significant in our calculation, since the values for λ 3000 Å have to be arbitrarily extrapolated. We have, therefore, set

$$\left. \begin{aligned} w_2(\nu) &= \Gamma A(\nu) & \text{for} & \quad \Gamma A(\nu) < 1 \\ &= 1 & \text{for} & \quad \Gamma A(\nu) > 1 \end{aligned} \right\} \quad (137)$$

Γ being a constant factor that takes into account the finite residual intensity of the lines. An inspection of the Utrecht *Atlas of the Solar Spectrum* indicates that the adop-

¹⁵ *Zs. f. A p.*, 11, 132, Table 2, 1935.

tion of the value $\Gamma = 2.5$ is reasonable. Once $w_2(\nu)$ is defined in numerical form, we can evaluate $\bar{w}_2(\tau)$ according to equation (105). For this purpose we have used tables of the function $B_\nu(T_r)/B(T_r)$ computed with the temperature distribution of a gray atmosphere in the fourth approximation, which were kindly provided by Dr. Chandrasekhar. The integrations were carried out graphically, and the resulting values of $\bar{w}_2(\tau)$ are given for small values of τ in Table 3. Two further constants have still to be fixed,

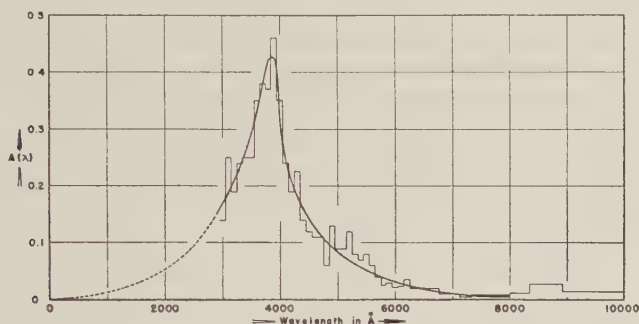


FIG. 3.—The distribution with wave length of the total absorption produced by the Fraunhofer lines in the solar spectrum.

TABLE 3

THE INTEGRATED PLANCK FUNCTION WHEN THE DISTRIBUTION
OF THE LINES DEPENDS ON FREQUENCY

τ	$\bar{w}_2(\tau)$	$\delta(\tau)$	$\frac{1}{F}B_1(\tau)$	$\Psi(\tau)$	$\frac{1}{F}B_2(\tau)$
0.00.....	0.0906	−0.0411	0.4034	0.00000	0.4296
0.05.....	.0998	−.0319	0.4530	.00028	0.4773
0.10.....	.1086	−.0231	0.5010	.00081	0.5235
0.15.....	.1167	−.0150	0.5477	.00155	0.5686
0.20.....	.1242	−.0075	0.5932	.00260	0.6136
0.30.....	.1372	+ .0055	0.6818	.00548	0.7039
0.40.....	.1405	+ .0178	0.7682	.00909	0.7929
0.50.....	.1602	+ .0285	0.8530	.01319	0.8819
0.60.....	.1690	+ .0373	0.9370	.01768	0.9719
0.80.....	.1842	+ .0525	1.1032	.02798	1.1544
1.00.....	.1979	+ .0662	1.2684	.03980	1.3388
1.40.....	.2213	+ .0896	1.5977	.06578	1.7070
1.80.....	.2404	+ .1087	1.9266	.09497	2.0802
2.20.....	.2543	+ .1226	2.2555	.12583	2.4831
3.00.....	0.2760	+0.1443	2.9133	0.19135	3.2252

namely, γ and ω_2 . Both of them can be fixed by trial and error in such a way that in the (1, 1) approximation we obtain the observed emergent flux in the lines and, at the same time, satisfy equation (131). We have found that the values $\omega_2 = 0.1317$ and $\gamma = 3$ satisfy these conditions. Actually, for the emergent flux we obtain $F^{(2)} = 0.0855 F$, against the observed value of $0.083 F$. Further, equation (132) is satisfied within 0.00003. The first approximation is therefore described by the constants

$$\omega_1 = 0.8683, \quad \omega_2 = 0.1317, \quad \gamma = 3, \quad (138)$$

which give for the temperature distribution

$$B_1(\tau) = 0.82219(\tau + 0.54331 - 0.05273e^{-k\tau}), \quad (139)$$

with

$$k = 4.41526. \quad (140)$$

The values of $B_1(\tau)$ for some values of τ are given in Table 3 and in Figure 1. We are now in a position to compute $B_2(\tau)$ according to formula (135), since, once ω_2 is known, the function $\delta(\tau)$ is fixed by equation (106). (Some of the values of $\delta(\tau)$ obtained in this manner are given in Table 3.) The definite integral appearing in expression (136) is then numerically evaluated, and the resulting values of $\Psi(\tau)$ are given in Table 3. Finally, the calculation of $B_2(\tau)$ according to equation (125) readily follows, and the values obtained are given in the last column of Table 3 and further illustrated in Figure 1.

How nearly the temperature distribution derived in this fashion represents the actual distribution of temperature in the outer layers of the sun is a question that depends more on the reliability of the data of observational character that we have used than on the approximations made to solve the problem of radiative equilibrium. In this context we may point out that if Mulders has underestimated the absorption caused by the lines to the violet of λ 4000 Å (and there is evidence to believe that such is the case) and if the absorption immediately to the violet of λ 3000 Å does not follow the course indicated in Figure 3, the values adopted for $w_2(\tau)$ would be systematically too low. An immediate consequence of higher values of ω_2 would be the lowering of the obtained boundary temperature, followed by a steeper temperature gradient. It would seem that with a further refinement of the theory, considering the dependence of the distribution function of the lines on depth, the temperature gradient for $\tau < 0.2$ will be steeper and will approach a situation similar to that described by the layer model. Nevertheless, we can consider that the boundary temperature provided by the model analyzed in this section gives an upper limit to the actual one. With $T_e = 5713^\circ$ K, the boundary temperature in the first approximation is 4550° K and 4630° K in the second—values which have to be compared with 4656° K given by the standard model.

7. *The distribution of temperature in the outer layers of the sun and the limb-darkening phenomenon.*—With the solution of the transfer problem completed in the preceding sections, we can now answer the question raised in the introductory section, namely, as to how the temperature distribution in the outer layers of the solar atmosphere is modified by the formation of absorption lines.

From an inspection of Figure 1, we can say, in the first place, that the effect of the Fraunhofer lines on the boundary temperature T_0 cannot be expected to increase the value given by the standard model. On the contrary, it is likely that the value of T_0 is somewhat lower than 4600° K, as was shown at the end of the preceding paragraph. This result is in substantial agreement with determinations of the "effective excitation temperature of the solar reversing layers" made by K. O. Wright¹⁶ and other authors.¹⁷ This shows clearly the misleading results given by the Milne blanketing model, which gives $T_0 = 5050^\circ$ K.¹⁸ In the layers in which the lines arise, that is, in the range $0 < \tau < 0.2$, the temperature increases rather fast compared to the standard model; but, for optical depths larger than 0.2, the temperature gradient has essentially the same value as in the standard case, the temperature itself being higher. Finally, around $\tau = 2$, where the convective equilibrium sets in, all our models will give a temperature gradient which is too high as compared with the real one. However, for a discussion of the emergent radiation, the existence of this zone of convective equilibrium can be disregarded, as Keenan¹⁹ has shown.

¹⁶ *Ap. J.*, **99**, 249, 1944.

¹⁷ R. B. King, *Ap. J.*, **87**, 40, 1938; Menzel, Baker, and Goldberg, *Ap. J.*, **87**, 81, 1938.

¹⁸ Cf. Mulders, *op. cit.*

¹⁹ *Ap. J.*, **87**, 45, 1938.

From the foregoing remarks it is now apparent that the temperature distribution in the solar atmosphere does not deviate greatly from the one obtained in the layer model considered in section 5.3 and that, therefore, the comparison of total darkening in this model with observations in integrated light on the solar disk is justified. From the comparison, carried out in Figure 2, we see that the agreement between observed and computed darkening is entirely satisfactory up to a distance of 0.95 (in units of the radius of the disk) from the center. However, for points still nearer the limb, the theoretical values are larger than the observed ones, as we should expect from the fact that the boundary temperature of the sun is lower than the one given by the layer model. Finally, this agreement between observed and theoretical integrated darkening suggests immediately the origin of the systematic departures found by the writer²⁰ between observed and computed monochromatic darkening. For, as we have seen, the darkening given by the gray atmosphere is too low, compared with the observed one, for two reasons: first, the temperature distribution of the sun differs from that of the gray body, and, second, the darkening measures have been made over wave-length intervals which include not only continuous spectrum but also some Fraunhofer lines. From the publications containing the original darkening measures by Abbot,²¹ it is not possible to find out how wide the wave-length regions were which were isolated by the spectrobolometer used to make the measurements; but there can be hardly any doubt that the measures for wave lengths below λ 6702 Å are affected by this cause.

From this discussion we may conclude that a definitive comparison between theoretical and observed monochromatic darkening in the sun can be made only when, on the theoretical side, we use a temperature distribution which takes into account the presence of absorption lines and the nongrayness of the atmosphere²² and when, on the observational side, observations made with higher resolving power are available.

I gratefully acknowledge the valuable suggestions I have received from Dr. S. Chandrasekhar in connection with the present work. I am also deeply indebted to Dr. Otto Struve for the facilities made available to me during my stay at the Yerkes Observatory.

²⁰ *Op. cit.*, see p. 391, Fig. 3.

²¹ *Ann. Smithsonian Inst.*, **3**, 153, 1913.

²² Cf. Paper VII, No. 6, p. 345.

